

Hill's formula

Sergey Bolotin*and Dmitry Treschev†

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Abstract

In his study of periodic orbits of the 3 body problem, Hill obtained a formula relating the characteristic polynomial of the monodromy matrix of a periodic orbit and an infinite determinant of the Hessian of the action functional. A mathematically correct definition of the Hill determinant and a proof of Hill's formula were obtained later by Poincaré. We give two multidimensional generalizations of Hill's formula: to discrete Lagrangian systems (symplectic twist maps) and continuous Lagrangian systems. We discuss additional aspects which appear in the presence of symmetries or reversibility. We also study the change of the Morse index of a periodic trajectory after the reduction of order in a system with symmetries. Applications are given to the problem of stability of periodic orbits.

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*Steklov Mathematical Institute, Moscow and UW-Madison, USA

†Steklov Mathematical Institute, Moscow and Lomonosov Moscow State University

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1 Introduction

In 1886, in his study of lunar orbits, Hill [1] discovered a formula which expresses the characteristic polynomial of the monodromy matrix for a second order time periodic

differential equation in terms of the determinant of a certain infinite matrix. Here is a slightly modified version of this result. Consider Hill's equation

$$\ddot{x} = a(t)x, \quad (1.1)$$

where

$$a(t) = \sum_{k=-\infty}^{+\infty} a_k e^{ikt}$$

is a real 2π -periodic function. Let ρ and ρ^{-1} be eigenfunctions of the monodromy matrix. Hill showed that

$$\frac{\rho + \rho^{-1} - 2}{e^{2\pi} + e^{-2\pi} - 2} = \det H, \quad (1.2)$$

where H is the infinite matrix¹

$$H = \left(\frac{k^2 \delta_{jk} + a_{k-j}}{k^2 + 1} \right)_{j,k \in \mathbb{Z}}, \quad (1.3)$$

and δ_{jk} is the Kronecker symbol.

Hill computed $\det H$ approximately replacing H by a 3×3 matrix, which gave quite a good approximation. He used equation (1.2) to find the multipliers approximately. Astronomical tables obtained by this method are well-known.

Hill's argument was not rigorous because he did not prove convergence for the infinite determinant $\det H$. Several years later Poincaré [2] explained an exact meaning of the Hill determinant and presented a rigorous proof of Hill's formula. Hill's result entered textbooks on differential equations, but was almost forgotten by dynamical systems community until the end of the XXth century when an analogue of equation (1.2) appeared for discrete Lagrangian systems in [3] and independently in [4]. Here H turned out to be the finite Hessian matrix associated with the action functional at the critical point generated by the periodic solution. In [5] (see also [6]) a general form of Hill's formula was obtained for a periodic solution of an arbitrary Lagrangian system on a manifold. In this case H is a properly regularized Hessian operator of the action functional at the critical point determined by a periodic solution.

Both discrete and continuous versions of Hill's formula give non-trivial information on the dynamical stability of the periodic orbit in terms of its Morse index. Recently this connection was investigated by means of symplectic geometry (see, for example, [7] and [8]). However, the approach based on the Hill determinant is sometimes simpler and provides additional insight to the problem.

As mentioned, there are two similar but formally different cases:

¹Hill's matrix was slightly different.

- Continuous Lagrangian system with configuration manifold M and τ -periodic Lagrangian $\mathcal{L}(x, \dot{x}, t)$ on $TM \times \mathbb{R}$ which is strictly convex in the velocity. Solutions of the Lagrangian system will be called trajectories. Then τ -periodic trajectories γ are critical points of the action functional

$$\mathcal{A}(\gamma) = \int_0^\tau \mathcal{L}(\gamma(t), \dot{\gamma}(t), t) dt$$

on the set of τ -periodic curves $\gamma: \mathbb{R} \rightarrow M$.

- Discrete Lagrangian system with Lagrangian $L(x, y)$ on $M \times M$ satisfying certain non-degeneracy condition. Then periodic trajectories are n -periodic sequences $\mathbf{x} = (x_i)_{i \in \mathbb{Z}}$ which are critical points of the action functional on M^n :

$$\mathcal{A}(\mathbf{x}) = \sum_{i=1}^n L(x_i, x_{i+1}), \quad x_{i+n} = x_i.$$

Usually one case can be reduced to the other, but this reduction may be cumbersome. Hence it makes sense to consider both cases separately.

Both versions of Hill's formula look similar. Let P be the monodromy matrix of the periodic trajectory, h the second variation of the action functional at the periodic trajectory, and H the corresponding Hessian operator. Then

$$\det(P - I) = \sigma(-1)^m \beta \det H, \quad (1.4)$$

where $m = \dim M$ and $\sigma = \pm 1$ takes care of orientation. The coefficient β is a positive scaling factor.

The operator H is self-adjoint in a proper Hilbert space. For continuous systems, H is an unbounded operator, so it needs to be regularized. For example, for Hill's equation (1.1), H is a Sturm–Liouville operator.

Another version of Hill's formula, a generalization of (1.4), has the form

$$\rho^{-m} \det(P - \rho I) = \sigma(-1)^m \beta \det H_\rho, \quad \rho \in \mathbb{C}, \quad (1.5)$$

where H_ρ is the ρ -Hessian which coincides with the ordinary Hessian for $\rho = 1$. It is self-adjoint if $|\rho| = 1$. Since P is symplectic, both sides of (1.5) are polynomials of degree m in $\rho + \rho^{-1}$.

Hill's formula (1.4) has many dynamical applications. The first one is the well known statement that the Poincaré degeneracy of a periodic trajectory (that is, the condition that 1 is an eigenvalue of P) is equivalent to the variational degeneracy (the condition $\det H = 0$).

Another application concerns dynamical instability of a periodic trajectory. It is based on the observation that the inequality $\det(P - I) < 0$ implies the existence of

a real multiplier (that is, an eigenvalue of P) $\rho > 1$. Indeed, $F(\rho) = \det(P - \rho I) = \det(\rho I - P) \rightarrow \infty$ as $\rho \rightarrow +\infty$, and so $F(1) < 0$ implies the existence of a root $\rho > 1$. Thus γ has a positive Lyapunov exponent and is exponentially unstable.

If $\det H \neq 0$, we have $\text{sign det } H = (-1)^{\text{ind } H}$, where $\text{ind } H$ is the Morse index of the periodic trajectory. Hence if the periodic trajectory is nondegenerate, then by (1.4) the inequality $\sigma(-1)^{m+\text{ind } H} < 0$ implies exponential instability in a ‘physical’ (with $\beta > 0$) system.

In some cases it is possible to prove that for any $|\rho| = 1$ the Hessian H_ρ is positive definite and therefore the equation $\det H_\rho = 0$ has no solutions on the unit circle. Then we obtain exponential instability, in fact, total hyperbolicity for the corresponding periodic trajectory (Propositions 4.5 and 7.2).

Below we present other dynamical consequences of Hill’s formula.

Note that the connections between dynamical and geometrical properties of periodic orbits are not restricted to Hill’s formula. We mention here interesting relations between stability properties and the structure (index, signature, and so on) of a quadratic first integral of the linearized system ([9] and [10]). Many interesting results follow from the index formula in symplectic geometry ([7] and [8]). Some of our results may be regarded as Lagrangian versions of the results of [11] and [12].

Hill’s formula is potentially most useful for the study of periodic orbits obtained by variational methods. Many such orbits were obtained recently in celestial mechanics by minimization of the action functional on appropriate classes of curves, see, for example, [13]–[15]. The most famous example is the figure eight orbit, see [13]. However, due to rotational and other symmetries, none of these periodic orbits are nondegenerate minimum points of the action.

In applications periodic trajectories are usually degenerate. For example, any periodic orbit of an autonomous continuous Lagrangian system is degenerate. In this case the variational equation has a τ -periodic solution, $\dot{\gamma}$, and a linear first integral, the linearization of the energy integral. Another reason for such a degeneracy (now in both discrete and continuous cases) is the presence of a symmetry group, preserving the Lagrangian. This degeneracy also gives τ -periodic solutions and linear integrals for the variational equation. For degenerate periodic trajectory equation (1.4) is useless because both sides vanish. A nondegenerate version of Hill’s formula can be obtained with the help of the reduction procedure. We consider the case when the Lie algebra V of symmetry vector fields for the variational equation is commutative, the dimension of the generalized unit eigenspace N of P is $2k$, where $k = \dim V$ (no further degeneracy) and a condition, called the non-degeneracy of the trajectory mod V , holds. The latter condition has a Lagrangian nature rather than Hamiltonian.

The reduced Hill’s formula looks similarly, but the corresponding monodromy and Hessian operators \tilde{P} and H^\perp act on smaller (reduced) spaces, and are nonde-

generate if all the symmetries are taken into account:

$$\det(\tilde{P} - I) = \sigma^\perp (-1)^{m-k} \beta^\perp \det H^\perp.$$

Here $\sigma^\perp \in \{1, -1\}$ is the ‘reduced orientation’ and $\beta^\perp > 0$.

Now an interesting question appears on the relation between σ and σ^\perp as well as between $\text{ind } H$ and $\text{ind } H^\perp$, the Morse indices of the Hessians in the original and reduced systems. Indeed, σ and $\text{ind } H$ are often known for solutions obtained by variational methods, while σ^\perp and $\text{ind } H^\perp$ appear in stability problems. The following construction explains our answer to this question.

Let h and h^\perp be bilinear second variation forms corresponding to the operators H and H^\perp respectively. The forms h and h^\perp are defined on the vector spaces X and X^\perp of variations along the periodic orbit, for the original and reduced Lagrangian system, respectively. The procedure of the order reduction gives a canonical projection $\Pi: X \rightarrow X^\perp$.

For any $\zeta \in V$ and $\eta \in X$ we have $h(\zeta, \eta) = 0$. Therefore h defines a bilinear form \hat{h} on $\hat{X} = X/V$ and $\text{ind } h = \text{ind } \hat{h}$. The spaces \hat{X} and X^\perp admit the expansions

$$\hat{X} = \hat{\Omega} \oplus \hat{Y}^0 \oplus \hat{Z}, \quad \hat{X}^\perp = \Omega^\perp \oplus Y^\perp$$

with the following properties:

- 1) $\dim \hat{\Omega} = \dim \Omega^\perp = k$;
- 2) the spaces $\hat{\Omega}$, \hat{Y}^0 , and \hat{Z} are \hat{h} -orthogonal, while Ω^\perp and Y^\perp are h^\perp -orthogonal;
- 3) the restriction $\hat{h}|_{\hat{Z}}$ is nondegenerate in the discrete case and positive definite in the continuous case;
- 4) $\Pi(\hat{Z}) = 0$, while the restrictions $\Pi|_{\hat{Y}^0}: \hat{Y}^0 \rightarrow Y^\perp$ and $\Pi|_{\hat{\Omega}}: \hat{\Omega} \rightarrow \Omega^\perp$ are linear isomorphisms;
- 5) the forms $\hat{h}|_{\hat{Y}^0}$ and $h^\perp|_{Y^\perp}$ coincide in the sense that $h^\top|_{Y^\perp} = h^\perp|_{Y^\perp}$, where $\hat{h} = h^\top \circ \Pi$;
- 6) $h^\top|_{\Omega^\perp} - h^\perp|_{\Omega^\perp} = \chi$, where χ is positive definite in the continuous case.

In a convenient basis we give an explicit expression for the matrices $h^\top|_{\Omega^\perp}$ and $h^\perp|_{\Omega^\perp}$. Using these expressions we show that

$$\sigma(-1)^{\text{ind } H} = \sigma^\perp(-1)^{\text{ind } H^\perp + \text{ind } b},$$

where the quadratic form b on the generalized eigenspace $N = \text{Ker}(P - I)^2$ is defined by $b(v) = \omega((P - I)v, v)$, where ω is the symplectic structure.

In some cases $(-1)^{\text{ind } b}$ has a clear dynamical meaning. For example, suppose that the degeneracy appears solely because the continuous Lagrangian is autonomous. Then $\dim V = 1$. The periodic trajectory γ belongs to a smooth family of periodic trajectories. Let E and τ be the energy and the period along this family. Then (see Lemma 6.4)

$$(-1)^{\text{ind } b} = -\text{sign } \frac{dE}{d\tau}.$$

Suppose that the periodic trajectory γ of an autonomous Lagrangian system has no other degeneracy. Then $k = 1$ and by the reduced Hill's formula it has a real multiplier $\rho > 1$ provided that

$$\sigma(-1)^{m+\text{ind } H} \frac{dE}{d\tau} < 0. \quad (1.6)$$

The sign of the quantity $dE/d\tau$ is easily computed, for example, in the problem of the motion of a point in \mathbb{R}^m in a homogeneous potential force field.

It turns out that closed geodesics do not satisfy the condition of non-degeneracy mod V . However we show that inequality (1.6) still implies the existence of a multiplier $\rho > 1$ provided no extra degeneracy takes place (Corollary 5.2). (Note that in this case $dE/d\tau < 0$).

As mentioned above, the subject of this paper is closely related to the theory of Maslov–Morse index for periodic orbits of Hamiltonian systems, see, for instance, [7], [8], [11], [12]). Some of our results can be obtained by these purely symplectic methods. Others are Lagrangian, and so do not have direct symplectic formulation. The situation is similar to the relation between Hamiltonian and Lagrangian systems: Hamiltonian theory is simpler, more general, and more powerful. Nevertheless for many problems the Lagrangian approach is essential.

This paper splits in two parts: discrete and continuous. Although the majority of constructions and statements in the discrete and continuous parts are analogous, there are many technical differences which forced us to deal with these two cases separately.

The plan of the paper is as follows. In § 2 we first recall the definition and basic properties of discrete Lagrangian systems (DLS). This material is well known to specialists, but these objects are not as standard as their continuous analogues.

Then we present several versions of Hill's formula for a periodic trajectory of a DLS. As an application, we give some sufficient conditions for the instability of periodic trajectories. Several statements concern stability problem for billiard systems in arbitrary dimension. For example, any n -periodic trajectory \mathbf{x} of a billiard system inside a hypersurface in \mathbb{R}^{m+1} such that $(-1)^{m+n+\text{ind } \mathbf{x}} < 0$ is exponentially unstable (Corollary 2.6). As far as we know, there are very few publications about stability of periodic trajectories in multidimensional DLS. Here we mention [16] and [17], where trajectories of period 2 are studied.

In § 3 we consider DLS with symmetry. We present a discrete version of Routh's procedure of order reduction and a reduced version of Hill's formula where the degeneracy which appears due to symmetry is removed. We also give a formula for the difference between the Morse index of a periodic trajectory of the original system and the Morse index of the corresponding periodic orbit of the reduced system.

In § 4 we study reversible DLS, that is, discrete Lagrangian systems with the Lagrangian L invariant under time reversal combined with an involution $S: M \rightarrow M$, $S^2 = \text{id}$. Thus $L(S(x), S(y)) = L(y, x)$ for any $x, y \in M$. For any trajectory $\mathbf{x} = (x_i)$ of the DLS, the sequence $\tilde{\mathbf{x}} = (Sx_{-i})$ is also a trajectory. If $\mathbf{x} = \tilde{\mathbf{x}}$ modulo a translation, the trajectory $\tilde{\mathbf{x}}$ is called S -reversible. Then the corresponding space of variation splits into a direct sum of spaces of odd and even variations with respect to S . Hill's determinant also admits splitting into a product of two determinants. Reversible periodic trajectories are also critical points of another action functional \mathcal{A}_+ which is obtained from the original one, \mathcal{A} , by restriction to the space of even variations. Morse index of a trajectory with respect to \mathcal{A}_+ is in general different from that computed with respect to \mathcal{A} .

Any S -reversible trajectory \mathbf{x} of a DLS has 0, 1, or 2 fixed points of the involution S . According to this we say that type $\mathbf{x} = 0, 1$, or 2 . One of application, presented in § 4 is as follows (Corollary 4.4). Suppose that the billiard surface $M \subset \mathbb{R}^{m+1}$ is symmetric relative to a hyperplane and S denotes this symmetry. Let \mathbf{x} be an S -reversible periodic billiard trajectory of type $\tau \in \{0, 1, 2\}$ which is a nondegenerate minimum of the 'half-length' \mathcal{A}_+ . If $m + \tau$ is odd, then \mathbf{x} is exponentially unstable.

In § 5 the continuous part of the paper starts. The main technical difference of the continuous case is the infinite dimension of the space of variations. Because of this the definition of Hill's determinant needs more care. We give a construction defining the Hill determinant and present several versions of Hill's formula analogous to the ones in the discrete case. Then we give applications to instability of periodic orbits of Lagrangian systems including the case of closed geodesics. A typical statement from this part (in fact, going back to Poincaré) is as follows. Let γ be a nondegenerate closed geodesic on an m -dimensional manifold and $\sigma(-1)^{m+\text{ind } \gamma} > 0$. Then γ is exponentially unstable (Corollary 5.3).

In § 6 we discuss the role of symmetries and give a version of Hill's formula which eliminates the corresponding degeneracy. Then we study the relation between the Morse index of the periodic trajectory of the original system and the corresponding periodic solution of the reduced system. We present some applications of this formula to the problem of stability for Lagrangian systems with symmetry.

Finally, in § 7 we consider a reversible CLS. The Lagrangian \mathcal{L} of an S -reversible NLS is compatible with the involution S in the following sense:

$$\mathcal{L}(S(x), dS(x)\dot{x}, t) = \mathcal{L}(x, -\dot{x}, -t).$$

As in the discrete case, the functional \mathcal{A}_+ corresponding to even variations is defined.

The main questions are the relation between the indices of an S -reversible periodic trajectory with respect to \mathcal{A} and \mathcal{A}_+ and the relation between the index with respect to \mathcal{A}_+ and stability properties. We show that in many cases the computation of $\text{ind } \gamma \bmod 2$ may be performed on variations from a $2m$ -dimensional space.

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2 Discrete case

2.1 Discrete Lagrangian systems (DLS)

Let M be an m -dimensional manifold and L a smooth² function on $M^2 = M \times M$. Denote

$$\partial_1 L(x, y) = \frac{\partial L(x, y)}{\partial x}, \quad \partial_2 L(x, y) = \frac{\partial L(x, y)}{\partial y} \quad (2.1)$$

and let

$$B(x, y) = -\partial_1 \partial_2 L(x, y).$$

In local coordinates,

$$B(x, y) = -\left(\frac{\partial^2 L}{\partial y_j \partial x_i} \right). \quad (2.2)$$

In invariant terms, $B(x, y)$ is a linear operator $T_x M \rightarrow T_y^* M$, or a bilinear form on $T_x M \times T_y M$. We say that L is a discrete Lagrangian if it satisfies the following condition.

Twist condition. $B(x, y)$ is nondegenerate for all $x, y \in M$.

Any discrete Lagrangian L locally defines a map

$$T: M^2 \rightarrow M^2, \quad T(x, y) = (y, z),$$

where $z = z(x, y)$ is determined by the equation

$$\frac{\partial}{\partial y} (L(x, y) + L(y, z)) = \partial_2 L(x, y) + \partial_1 L(y, z) = 0. \quad (2.3)$$

In general, T is a multivalued map (relation) with the graph

$$\Gamma = \{(x, y, y, z) \in M^2 \times M^2 : \partial_2 L(x, y) + \partial_1 L(y, z) = 0\}.$$

The dynamical system determined by T is called the discrete Lagrangian system (DLS) with configuration space M and Lagrangian L .

²Actually, C^2 is enough.

Remark 2.1. In this paper we deal with a small neighbourhood of a periodic orbit. Hence it is sufficient to assume that the non-degeneracy condition holds locally.

It is easy to check (see, for example, [18]) that T is symplectic with respect to the symplectic 2-form $\omega = B(x, y) dx \wedge dy$,

$$\omega(\mathbf{u}, \mathbf{v}) = \langle B(x, y)u_1, v_2 \rangle - \langle B(x, y)v_1, u_2 \rangle, \quad \mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2) \quad (2.4)$$

($\langle \cdot, \cdot \rangle$ is the canonical pairing of a covector on a vector).

Remark 2.2. Let us pass to Hamiltonian variables by the map $S: M^2 \rightarrow T^*M$, $(x, y) \mapsto (x, p_x)$, $p_x = -\partial_1 L(x, y)$. It is locally invertible and replaces T by a locally defined map $F = STS^{-1}: T^*M \rightarrow T^*M$. The map F is symplectic with respect to the standard symplectic form $dp_x \wedge dx$ on T^*M , and L is the generating function of F :

$$F(x, p_x) = (y, p_y), \quad p_x = -\partial_1 L(x, y), \quad p_y = \partial_2 L(x, y).$$

Such a symplectic map F is usually called a twist map.

The map T remains the same after multiplication of the Lagrangian by a constant, after addition of a constant to L , and after the so-called gauge transformation

$$L(x, y) \mapsto L(x, y) + f(x) - f(y)$$

with an arbitrary smooth function f on M .

A typical example of DLS is the multidimensional standard map:

$$L(x, y) = \frac{1}{2} \langle B(x - y), x - y \rangle - \frac{1}{2} (V(x) + V(y)), \quad x, y \in \mathbb{R}^m, \quad (2.5)$$

where B is a symmetric constant nondegenerate matrix.³

Consider a domain in \mathbb{R}^{m+1} bounded by a smooth convex hypersurface M . The billiard system is a DLS with the Lagrangian $L(x, y) = |x - y|$ on $M \times M$. Let $\langle B(x, y)v, w \rangle$ be the bilinear form on $T_x M \times T_y M$ corresponding to the operator $B(x, y): T_x M \rightarrow T_y^* M$. A computation gives

$$\langle B(x, y)v, w \rangle = \frac{\langle v, w \rangle - \langle v, e \rangle \langle w, e \rangle}{|x - y|}, \quad e = \frac{x - y}{|x - y|}. \quad (2.6)$$

We may identify $T_x M$ and $T_y M$ by an isomorphism $\Pi(x, y): T_x M \rightarrow T_y M$, which is the parallel projection in \mathbb{R}^{m+1} along the segment $[x, y]$: $\Pi v = v \pmod{e}$. Then

$$\langle B(x, y)v, \Pi(x, y)v \rangle = \frac{|v|^2 - \langle v, e \rangle^2}{|x - y|} > 0, \quad v \in T_x M \setminus \{0\}.$$

We orient M as the boundary. Since $\Pi(x, y)$ changes orientation, we obtain

³One can replace the potential $(V(x) + V(y))/2$ by $V(x)$ or $V(y)$ because they are all gauge-equivalent.

Proposition 2.1. $\det B(x, y) < 0$.

Since the image and the range of $B(x, y)$ are different, $\det B(x, y)$ is not invariantly defined, but its sign is. The fact that the map $B(x, y)$ is nondegenerate, provided the hyperplanes $T_x M$ and $T_y M$ are not parallel to each other in \mathbb{R}^m , is well-known; for a recent reference see [19].

In [18] the reader can find many examples of (mostly integrable) DLS, including multidimensional ones.

For a continuous Lagrangian system (CLS) with Lagrangian $\mathcal{L}(x, \dot{x})$, an analogue of the operator $B(x, y)$ is the matrix $\mathcal{L}_{\dot{x}\dot{x}}(x, \dot{x})$ of second partial derivatives. Indeed, consider a DLS on \mathbb{R}^m with the Lagrangian $L(x, y) = \mathcal{L}(x, (y - x)/\varepsilon)$. In the limit as $\varepsilon \rightarrow 0$, orbits of DLS converge to orbits of the CLS with the Lagrangian \mathcal{L} . A computation shows that

$$\varepsilon^2 B(x, y) = \mathcal{L}_{\dot{x}\dot{x}}(x, (y - x)/\varepsilon) + O(\varepsilon).$$

In particular, for an analogue of a positive definite Lagrangian system, $\det B(x, y) > 0$.

For $m \geq 2$ there is no universally accepted discrete analogue of positive definite continuous Lagrangian systems. Indeed, in general B is not symmetric and, moreover, its symmetry does not have an invariant meaning since B and B^* are defined on different spaces. Note that the 1-dimensional Aubry–Mather theory was developed for twist maps, while multidimensional theory is well developed for continuous positive definite Lagrangian systems.

The most common definition of a positive definite DLS is as follows. Let $M = \mathbb{R}^m$ and suppose L satisfies the following conditions (see, for example, [20]):

- the function $L(x, x + v)$ is periodic in $x \in \mathbb{T}^m$ and superlinear in $v \in \mathbb{R}^m$;
- for any $x \in \mathbb{R}^m$ the map $y \mapsto \partial_1 L(x, y)$ is a diffeomorphism of \mathbb{R}^m .

Then L is a generating function of a globally defined symplectic twist map of $\mathbb{T}^m \times \mathbb{R}^m$. Evidently, such L satisfies $\det B(x, y) > 0$.

2.2 Discrete Hill determinant

Let $(x_i)_{i \in \mathbb{Z}}$, $x_{i+n} = x_i$, be an n -periodic trajectory of a DLS, that is, $T(x_{i-1}, x_i) = (x_i, x_{i+1})$ for all i . The periodic orbit is determined by $\mathbf{x} = (x_1, \dots, x_n) \in M^n$, and a cyclic permutation of \mathbf{x} gives the same orbit. By (2.3),

$$\partial_2 L(x_{i-1}, x_i) + \partial_1 L(x_i, x_{i+1}) = 0, \quad i = 1, \dots, n, \quad (2.7)$$

where $x_0 = x_n$ and $x_1 = x_{n+1}$. Thus, \mathbf{x} is a critical point of the action functional

$$\mathcal{A}(\mathbf{x}) = L(x_1, x_2) + L(x_2, x_3) + \cdots + L(x_n, x_1), \quad \mathbf{x} \in M^n.$$

The point $p = (x_1, x_2)$ is a fixed point of the map $T^n: M^2 \rightarrow M^2$. The linear approximation to dynamics of T near the periodic trajectory is determined by the linear Poincaré map $P = DT^n(p): W \rightarrow W$, $W = T_p M^2$. In local coordinates, P becomes the monodromy matrix defined uniquely up to a similarity $P \mapsto S^{-1}PS$. Eigenvalues of P are called multipliers of the periodic orbit. They determine dynamical properties of the periodic trajectory in the linear approximation.

Let

$$\mathbf{H} = \frac{\partial^2 \mathcal{A}(\mathbf{x})}{\partial \mathbf{x}^2}$$

be the Hessian matrix of \mathcal{A} at the critical point \mathbf{x} . Denote

$$B_i = B(x_i, x_{i+1}), \quad x_{n+1} = x_1.$$

Theorem 2.1 (discrete Hill formula).

$$\det(P - I) = \frac{(-1)^m \det \mathbf{H}}{\prod_{i=1}^n \det B_i} = \sigma(-1)^m \beta \det \mathbf{H}, \quad (2.8)$$

$$\sigma(\mathbf{x}) = \text{sign} \prod_{i=1}^n \det B_i, \quad \beta = \left| \prod_{i=1}^n \det B_i \right|^{-1}. \quad (2.9)$$

For ‘physical’ discrete Lagrangians the geometrical meaning of σ is the orientability: the trajectory \mathbf{x} is, in a certain sense, orientable if $\sigma(\mathbf{x}) > 0$ and non-orientable otherwise. For example, this is true if DLS is obtained by discretization of a positive definite CLS. By Proposition 2.1, for a billiard n -periodic trajectory \mathbf{x} $\sigma(\mathbf{x}) = (-1)^n$. Therefore in this sense billiard periodic trajectories with odd period are non-orientable. Note that σ is replaced by $(-1)^{mn}\sigma$ if we replace L by $-L$.

2.3 Invariant meaning of Hill’s formula

The left-hand side of (2.8) obviously does not depend on the choice of local coordinates in M . However an invariant meaning of the right hand side is a priori not clear. Let us explain why it is coordinate independent. Let $E_i = T_{x_i} M$. Then $B_i = B(x_i, x_{i+1})$ is a linear operator $E_i \rightarrow E_{i+1}^*$, and

$$A_i = \partial_{22} L(x_{i-1}, x_i) + \partial_{11} L(x_i, x_{i+1})$$

is a symmetric operator $A_i: E_i \rightarrow E_i^*$.

The Hessian of \mathcal{A} at the critical point $\mathbf{x} \in M^n$ is a symmetric bilinear form h on $X = T_{\mathbf{x}}M^n = E_1 \times \cdots \times E_n$ given by

$$h(\mathbf{u}, \mathbf{v}) = \sum_{i=1}^n (\langle A_i u_i, v_i \rangle - \langle B_{i-1} u_{i-1}, v_i \rangle - \langle B_i^* u_{i+1}, v_i \rangle), \quad (2.10)$$

where

$$\mathbf{u} = (u_1, \dots, u_n), \quad \mathbf{v} = (v_1, \dots, v_n), \quad u_0 = u_n, \quad u_{n+1} = u_1.$$

The form h is represented by a symmetric operator $\mathbf{H}: X \rightarrow X^*$:

$$h(\mathbf{u}, \mathbf{v}) = \langle \mathbf{H}\mathbf{u}, \mathbf{v} \rangle, \quad \mathbf{u}, \mathbf{v} \in X,$$

where

$$(\mathbf{H}\mathbf{u})_i = A_i u_i - B_{i-1} u_{i-1} - B_i^* u_{i+1}, \quad i = 1, \dots, n.$$

Define linear operators $\mathbf{A}, \mathbf{B}: X \rightarrow X^*$ by

$$(\mathbf{A}\mathbf{u})_i = A_i u_i, \quad (\mathbf{B}\mathbf{u})_i = -B_{i-1} u_{i-1}, \quad (\mathbf{B}^* \mathbf{u})_i = -B_i^* u_{i+1}. \quad (2.11)$$

Then

$$\mathbf{H} = \mathbf{A} + \mathbf{B} + \mathbf{B}^*.$$

Since the B_i are nondegenerate, \mathbf{B} is invertible. If we introduce local coordinates, then \mathbf{B} becomes an $(mn \times mn)$ -matrix, and

$$\det \mathbf{B} = (-1)^m \prod_{i=1}^n \det B_i. \quad (2.12)$$

Hence Hill's formula takes the invariant form⁴

$$\det(P - I) = \frac{\det \mathbf{H}}{\det \mathbf{B}} = \det(\mathbf{B}^{-1} \mathbf{H}). \quad (2.13)$$

The equation $\mathbf{H}\mathbf{u} = 0$ gives the variational system of the periodic trajectory \mathbf{x} :

$$A_i u_i - B_{i-1} u_{i-1} - B_i^* u_{i+1} = 0, \quad u_i \in E_i, \quad i \in \mathbb{Z}. \quad (2.14)$$

This is the linear approximation to the system (2.7) near the periodic trajectory \mathbf{x} . More precisely, if u_i is any solution of the variational system, then the linearized map $P_i = dT(x_{i-1}, x_i)$ acts as

$$P_i(u_{i-1}, u_i) = (u_i, u_{i+1}), \quad u_{i+1} = (B_i^*)^{-1}(A_i u_i - B_{i-1} u_{i-1}).$$

The kernel of $\mathbf{H}: X \rightarrow X^*$ is the set of n -periodic solutions $\mathbf{v} = (v_i)$, $v_{i+n} = v_i$, of (2.14).

The variational system is a linear Lagrangian system.

⁴Since $\det \mathbf{H}, \det \mathbf{B}$ are linear operators of 1-dimensional spaces $\wedge^{mn} X \rightarrow \wedge^{mn} X^*$, their quotient is a well defined scalar.

Definition 2.1. A linear periodic discrete Lagrangian system (E, Λ) is defined by n -periodic sequences $E = (E_i)_{i \in \mathbb{Z}}$ of vector spaces and linear operators $A_i: E_i \rightarrow E_i^*$, $B_i: E_i \rightarrow E_{i+1}^*$, where A_i is symmetric and B_i is nondegenerate. The Lagrangian is

$$\Lambda_i(u_i, u_{i+1}) = \frac{1}{2} \langle A_i u_i, u_i \rangle - \langle B_i u_i, u_{i+1} \rangle. \quad (2.15)$$

Trajectories of (E, Λ) are sequences $\mathbf{u} = (u_i)$ such that

$$\partial_{u_i}(\Lambda_{i-1}(u_{i-1}, u_i) + \Lambda_i(u_i, u_{i+1})) = 0.$$

Thus trajectories of (E, Λ) satisfy the variational system (2.14) and are extremals of the quadratic action functional

$$\frac{1}{2} h(\mathbf{u}, \mathbf{u}) = \sum_{i=1}^n \Lambda_i(u_i, u_{i+1}). \quad (2.16)$$

The system (E, Λ) is the linearization of (M, L) at \mathbf{x} .

2.4 Generalized Hill determinant

Let us define a generalization of the Hessian \mathbf{H} . Let $S^1 = \{\rho \in \mathbb{C} : |\rho| = 1\}$. For any $\rho \in S^1$, let X_ρ be the space of all quasiperiodic complex vector sequences $\mathbf{u} = (u_j)_{j \in \mathbb{Z}}$ such that $u_{j+n} = \rho u_j$. Here u_j lies in the complexification of E_j which we will denote E_j for simplicity. The Hessian of the action defines a Hermitian form on X_ρ :

$$h(\mathbf{u}, \bar{\mathbf{v}}) = \sum_{j=1}^n (\langle A_j u_j, \bar{v}_j \rangle - \langle B_{j-1} u_{j-1}, \bar{v}_j \rangle - \langle B_j^* u_{j+1}, \bar{v}_j \rangle).$$

Since a quasiperiodic sequence $(u_i)_{i \in \mathbb{Z}}$ is determined by $\mathbf{u} = (u_1, \dots, u_n) \in E_1 \times \dots \times E_n = X$, we identify X_ρ with X (more precisely, with the complexification of X). Then we obtain the Hermitian form

$$h_\rho(\mathbf{u}, \bar{\mathbf{v}}) = \langle \mathbf{H}_\rho \mathbf{u}, \bar{\mathbf{v}} \rangle, \quad \mathbf{u}, \mathbf{v} \in X,$$

where $\mathbf{H}_\rho: X \rightarrow X^*$ is given by

$$\begin{aligned} (\mathbf{H}_\rho \mathbf{u})_j &= A_j u_j - B_{j-1} u_{j-1} - B_j^* u_{j+1}, \\ j &= 1, \dots, n, \quad u_0 = \rho^{-1} u_n, \quad u_{n+1} = \rho u_1. \end{aligned}$$

Similarly we define an operator $\mathbf{B}_\rho: X \rightarrow X^*$:

$$(\mathbf{B}_\rho \mathbf{u})_j = -B_{j-1} u_{j-1}, \quad j = 1, \dots, n, \quad u_0 = \rho^{-1} u_n.$$

Then

$$\mathbf{H}_\rho = \mathbf{A} + \mathbf{B}_\rho + \mathbf{B}_\rho^*.$$

The operators \mathbf{H}_ρ , \mathbf{B}_ρ make sense for any non-zero $\rho \in \mathbb{C}$.

Theorem 2.2 (generalized Hill formula). *For any $\rho \in \mathbb{C}$*

$$\det(P - \rho I) = \frac{\det \mathbf{H}_\rho}{\det \mathbf{B}_\rho}. \quad (2.17)$$

Since $\det \mathbf{B}_\rho = \rho^{-m} \det \mathbf{B}$, we obtain

$$\rho^{-m} \det(P - \rho I) = \frac{\det \mathbf{H}_\rho}{\det \mathbf{B}}. \quad (2.18)$$

This is an invariant version of the result of [3]. For $\rho = 1$, (2.18) gives (2.8).

Both sides in (2.18) are polynomials of degree m in the Hill *discriminant* $\rho + \rho^{-1}$ with senior coefficient 1. Indeed, the characteristic polynomial $F(\rho) = \det(P - \rho I)$ of the symplectic operator P satisfies $F(\rho) = \rho^{2m} F(\rho^{-1})$. Hence $\rho^{-m} F(\rho)$ is a symmetric polynomial in ρ and ρ^{-1} . Thus it is a function of $\rho + \rho^{-1}$.

In coordinates, \mathbf{H}_ρ is an $(mn \times mn)$ -matrix which coincides with \mathbf{H} with two exceptions: in the upper right $(m \times m)$ -block, $-B_n$ is replaced by $-\rho^{-1}B_n$ and in the lower left $(m \times m)$ -block, $-B_n^*$ is replaced by $-\rho B_n^*$.

Proof Theorem 2.2. Let us show that $G(\rho) = \det(\mathbf{B}_\rho^{-1} \mathbf{H}_\rho)$ is a polynomial of degree $2m$ with senior coefficient equal to 1.

Let $\nu = \rho^{1/n}$ and make a change of variables $\mathbf{u} \mapsto \mathbf{w}$, $u_j = \nu^j w_j$. Then the operator \mathbf{H}_ρ is replaced by $\widehat{\mathbf{H}}_\rho: X \rightarrow X^*$, where

$$(\widehat{\mathbf{H}}_\rho \mathbf{w})_j = A_j w_j - \nu^{-1} B_{j-1} w_{j-1} - \nu B_j^* w_{j+1}, \quad w_n = w_0, \quad w_{n+1} = w_1.$$

Hence

$$\widehat{\mathbf{H}}_\rho = \mathbf{A} + \nu^{-1} \mathbf{B} + \nu \mathbf{B}^*.$$

Similarly, \mathbf{B}_ρ is replaced by $\widehat{\mathbf{B}}_\rho = \nu^{-1} \mathbf{B}$. By the invariance of a determinant,

$$G(\rho) = \det(\widehat{\mathbf{B}}_\rho^{-1} \widehat{\mathbf{H}}_\rho) = \det(\nu \mathbf{B}^{-1} \mathbf{A} + I + \nu^2 \mathbf{B}^{-1} \mathbf{B}^*)$$

is a polynomial of order $2mn$ in ν . Since \mathbf{B}_ρ and \mathbf{H}_ρ are linear in ρ and ρ^{-1} , $G(\rho)$ is a polynomial in ρ , ρ^{-1} . Thus, $G(\rho)$ is a polynomial of order $2m$ in ρ . The senior coefficient is $\det(\mathbf{B}^{-1} \mathbf{B}^*) = 1$.

We have $\mathbf{H}_\rho \mathbf{u} = 0$ if and only if \mathbf{u} satisfies the variational system and $u_n = \rho u_0$, $u_{n+1} = \rho u_1$, or $Pw = \rho w$, where $w = (u_0, u_1)$. Hence $F(\rho) = \det(P - \rho I) = 0$ is equivalent to $\det \mathbf{H}_\rho = 0$. Thus the polynomials $F(\rho)$ and $G(\rho)$ have the same roots and so they coincide. \square

2.5 Some applications

Identity (2.8) implies that dynamical non-degeneracy of a periodic trajectory is equivalent to the geometric non-degeneracy: $\det \mathbf{H} = 0 \Leftrightarrow \det(P - I) = 0$. Actually, the proof of Theorem 2.2 was based on this fact.

Equation (2.8) gives

$$\sigma(\mathbf{x})(-1)^m \det \mathbf{H} \det(P - I) > 0.$$

Corollary 2.1. *Suppose that $\sigma(\mathbf{x})(-1)^{m+\text{ind } \mathbf{H}} < 0$. Then \mathbf{x} is dynamically unstable: there is a real multiplier $\rho > 1$.*

For example, the hypothesis holds when $\sigma(\mathbf{x})(-1)^m < 0$ and \mathbf{x} is a nondegenerate local minimum of the action \mathcal{A} .

Corollary 2.2. *Suppose that $\sigma(\mathbf{x})(-1)^{m+n} < 0$ and \mathbf{x} is a nondegenerate local maximum of the action \mathcal{A} . Then \mathbf{x} has a real multiplier $\rho > 1$.*

Indeed, it is sufficient to use the following

Proposition 2.2. *If $\det(P - I) < 0$, there is a real positive multiplier $\rho > 1$.*

Proof. Consider the characteristic polynomial $F(\rho) = \det(P - \rho I)$. Its roots are the multipliers of the periodic solution \mathbf{x} . We have $F(+\infty) = +\infty$ and $F(1) = \det(P - I) < 0$. Then there exists a real root $\rho > 1$. \square

Corollary 2.3. *If $\sigma(\mathbf{x})(-1)^{\text{ind } \mathbf{H}_{-1}} < 0$, there is a real positive multiplier $\rho < -1$.*

Indeed, for $\rho = -1$ Hill's formula (2.18) gives

$$\sigma(\mathbf{x}) \det \mathbf{H}_{-1} \det(I + P) > 0.$$

Let \mathbf{x}^2 be the iterate of a periodic trajectory \mathbf{x} , that is, the corresponding $2n$ -periodic trajectory.

Corollary 2.4. *Suppose $\sigma(\mathbf{x})(-1)^{\text{ind } \mathbf{H}(\mathbf{x}^2) - \text{ind } \mathbf{H}(\mathbf{x})} < 0$. Then \mathbf{x} is exponentially unstable.*

Proof. Since $2n$ -periodic vector fields along \mathbf{x}^2 are split into n -periodic and n -antiperiodic ones, $\text{ind } \mathbf{H}(\mathbf{x}^2) = \text{ind } \mathbf{H}(\mathbf{x}) + \text{ind } \mathbf{H}_{-1}(\mathbf{x})$. It remains to use Corollary 2.3. \square

In the case $m = 1$ there is a possibility to identify hyperbolicity or ellipticity of a periodic trajectory in terms of the index.

Corollary 2.5. *Suppose that $m = 1$. Then a nondegenerate periodic trajectory \mathbf{x} is hyperbolic if and only if $\text{ind } \mathbf{x}^2$ is even and elliptic if and only if $\text{ind } \mathbf{x}^2$ is odd.*

Proof. The hyperbolicity of \mathbf{x} is equivalent to the hyperbolicity of \mathbf{x}^2 . For $m = 1$, \mathbf{x}^2 is hyperbolic if and only if it has a multiplier $\rho > 1$. This is equivalent to the inequality $\sigma(\mathbf{x}^2)(-1)^{1+\text{ind } \mathbf{H}(\mathbf{x}^2)} < 0$. It remains to note that $\sigma(\mathbf{x}^2) = \sigma^2(\mathbf{x}) > 0$. \square

Consider the convex billiard bounded by a hypersurface M in \mathbb{R}^{n+1} . Then the corresponding action is length and, by Proposition 2.1, $\det B(x, y) > 0$. Therefore, $\sigma = (-1)^n$ and we obtain

Corollary 2.6. *Suppose $(-1)^{m+n+\text{ind } \mathbf{H}(\mathbf{x})} < 0$. Then \mathbf{x} is exponentially unstable by Corollary 2.1.*

In particular, \mathbf{x} is exponentially unstable in each of the following two cases

- if m is odd and \mathbf{x} is a nondegenerate local maximum of the billiard length functional;
- if $m + n$ is odd and \mathbf{x} is a nondegenerate local minimum of the billiard length functional.

For $m = 1$ by the Birkhoff theorem [21] (see also [6]), any convex billiard system has (at least) two periodic trajectories of period n with rotation number $k < n$, where one of them is a maximum of length, and hence generically hyperbolic. The other has index 1, and so $\det(P - I) > 0$. This implies that the trajectory has no real multipliers > 1 . Indeed, if such a multiplier exists, then the other one is also real and greater than 1. This contradicts $\det P = 1$.

The problem of stability for billiard trajectories of period 2 is systematically studied in the recent paper [17]. In this case, the characteristic polynomial, as a function of $\rho + \rho^{-1}$, can be presented as a determinant of some $m \times m$ matrix. This matrix is explicitly determined by the matrices of second fundamental forms of the surface M at the end points of the trajectory.

The requirement for n to be even in the hyperbolicity condition for a periodic trajectory of minimal length ($m = 1$) at the first glance looks somewhat strange because the billiard trajectory minimizing \mathcal{A} is naturally associated with a locally shortest closed geodesic on a two-dimensional Riemannian manifold. Such geodesics due to Poincaré [2] are known to be hyperbolic. However one should keep in mind that this Poincaré's result is valid only for orientable geodesics (see details in §5) while a periodic billiard trajectory with an odd period should be associated with a non-orientable geodesic.

A simple example of an elliptic action minimizing billiard trajectory with odd period can be constructed as follows. Let the billiard curve be an acute-angled triangle ABC . Then by a well-known theorem from planimetry the projections A' , B' , and C' of the vertices to the opposite sides form a triangle (the orthotriangle) which presents a local nondegenerate minimum of the billiard action (Fig. 1, a).

The corresponding periodic trajectory is parabolic: its multipliers $\rho_{1,2}$ are equal to -1 .

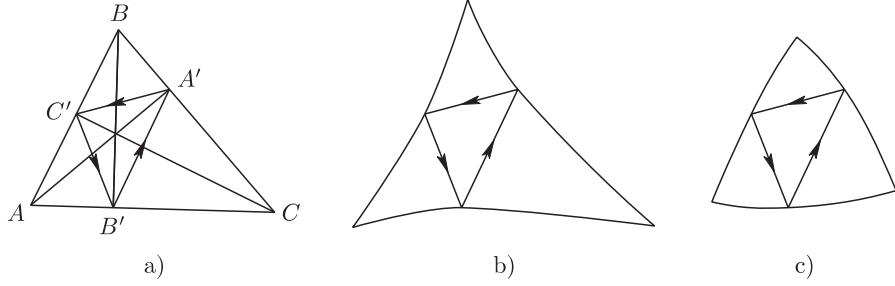


Figure 1: Deformation of a parabolic periodic orbit

A small deformation of the billiard curve does not destroy the periodic trajectory $A'B'C'$ and just slightly deforms it. If the boundary curve becomes concave, we obtain a Sinai billiard [22]. In this case the trajectory is hyperbolic (Fig. 1, b)). If the boundary curve becomes strictly convex (the curvature gets positive (Fig. 1, c)), then the trajectory becomes elliptic still having a locally minimal action provided the deformation is small.

3 Continuous symmetry in a DLS

3.1 Discrete symmetry

A diffeomorphism $\psi: M \rightarrow M$ is a discrete *symmetry* of the Lagrangian L if the map $\tilde{\psi} = \psi \times \psi: M^2 \rightarrow M^2$, $\tilde{\psi}(x, y) = (\psi(x), \psi(y))$ preserves L :

$$L(\psi(x), \psi(y)) = L(x, y).$$

A more general definition is that ψ preserves L up to a cocycle:

$$L(\psi(x), \psi(y)) = L(x, y) + f(x) - f(y).$$

If there exists a function g such that $g \circ \psi - g = f$, then ψ preserves the gauge-equivalent Lagrangian $\hat{L}(x, y) = L(x, y) + g(x) - g(y)$.

Proposition 3.1. *A symmetry takes a trajectory of a DLS into a trajectory. Thus, $\tilde{\psi} \circ T = T \circ \tilde{\psi}$.*

Proof. Since ψ preserves the action functional \mathcal{A} , it takes critical points to critical points. \square

If a DLS (M, L) admits a discrete symmetry group Γ , then, in principle, symmetry can be removed by a factorization $\widetilde{M} = M/\Gamma$ of the configuration space. However it is useful to keep in mind the following two aspects.

1. Since in general $L(x, y) \neq L(g_1 x, g_2 y)$ for $g_1 \neq g_2$, the Lagrangian becomes multivalued after the factorization M/Γ . This phenomenon is effectively used in the construction of a symbolic dynamics by the method of anti-integrable limit, see [23] and [24] (a more general setup is discussed in [25], a continuous analogue is presented in [26]).

2. A periodic trajectory of the original system can turn into a trajectory of the factorized system with a smaller period. Therefore the trajectory can lose orientability. Moreover, the configuration space itself can lose orientability. This happens, for example, in the case of a billiard system inside a convex hypersurface $M \subset \mathbb{R}^{m+1}$ symmetric with respect to the origin. Then $G = \{\text{id}, S\}$, where $S(x) = -x$. Then $\widetilde{M} = M/G$ is homeomorphic to the m -dimensional projective space which is non-orientable for m even.

3.2 Noether symmetry

Let \mathbf{w} be a smooth vector field on the configuration space M and $\psi_s: M \rightarrow M$ its phase flow. We say that \mathbf{w} is a *symmetry field* for the DLS if ψ_s is a symmetry for L for all s .

Define the vector fields $\mathbf{w}_1 = (\mathbf{w}, 0)$ and $\mathbf{w}_2 = (0, \mathbf{w})$ on M^2 . Let $\widetilde{\mathbf{w}} = \mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{w}, \mathbf{w})$ be the vector field corresponding to the group action $\widetilde{\psi}_s = \psi_s \times \psi_s$. We have an equivalent version of the definition: \mathbf{w} is a symmetry field for L if and only if

$$D_{\widetilde{\mathbf{w}}}L(x, y) = D_{\mathbf{w}}f(x) - D_{\mathbf{w}}f(y) \quad (3.1)$$

for some function f on M . If we replace L by its proper calibration, equation (3.1) can be replaced by

$$D_{\widetilde{\mathbf{w}}}L = D_{\mathbf{w}_1}L + D_{\mathbf{w}_2}L = 0. \quad (3.2)$$

Proposition 3.2. *Let \mathbf{w} be a symmetry field for L . Then*

$$\mathcal{J} = D_{\mathbf{w}_1}L = -D_{\mathbf{w}_2}L$$

is a first integral of the corresponding DLS, that is, $\mathcal{J} \circ T = \mathcal{J}$.

Proof. Suppose that $(y, z) = T(x, y)$. Then

$$\begin{aligned} \mathcal{J}(x, y) - \mathcal{J}(y, z) &= D_{\mathbf{w}_2}L(y, z) - D_{\mathbf{w}_2}L(x, y) \\ &= D_{\mathbf{w}_2}L(y, z) + D_{\mathbf{w}_1}L(y, z) = D_{\widetilde{\mathbf{w}}}L(y, z) = 0. \end{aligned}$$

The last expression vanishes by (3.2). □

We call \mathcal{J} the Noether integral.

Proposition 3.3. *Let \mathbf{w} be a symmetry field. Then the group action $\tilde{\psi}_s: M^2 \rightarrow M^2$ preserves \mathcal{J} . Equivalently, $\tilde{\mathbf{w}}$ is tangent to the level surfaces*

$$N_c = \mathcal{J}^{-1}(c) \subset M^2.$$

Proof. The derivative of \mathcal{J} along $\tilde{\mathbf{w}}$ is

$$D_{\tilde{\mathbf{w}}} \mathcal{J} = (D_{\mathbf{w}_1} + D_{\mathbf{w}_2}) D_{\mathbf{w}_1} L = 0$$

by (3.2) because the differential operators $D_{\mathbf{w}_1}$ and $D_{\mathbf{w}_1} + D_{\mathbf{w}_2}$ commute. \square

Note that \mathcal{J} is the Hamiltonian generating the group $\tilde{\psi}_s$ of symplectic transformations with respect to the symplectic form (2.4) on M^2 .

3.3 Routh reduction of order

Suppose system (M, L) admits commuting independent symmetry fields $\mathbf{w}^1, \dots, \mathbf{w}^k$:

$$[\mathbf{w}^\alpha, \mathbf{w}^\beta] = 0, \quad \alpha, \beta = 1, \dots, k.$$

Then the flows $\psi_s^\alpha: M \rightarrow M$ of symmetry fields commute. Let G be the corresponding commutative group acting on M by

$$x \mapsto \psi_s(x) = \psi_{s_1}^1 \circ \dots \circ \psi_{s_k}^k(x), \quad s \in \mathbb{R}^k.$$

In general the flows $\psi_{s_\alpha}^\alpha$ may be incomplete, and then G is a local group acting on M . Since we are interested in a neighbourhood of a periodic orbit, these non-local questions are irrelevant for us. Suppose that $\tilde{M} = M/G$ is a smooth manifold and $\pi: M \rightarrow \tilde{M}$ a smooth fibration (at least locally this is always true).

Let \mathcal{J}^α be the Noether integral of (M, L) corresponding to \mathbf{w}^α and let $\mathcal{J} = (\mathcal{J}^1, \dots, \mathcal{J}^k)$ be the corresponding vector integral. We fix the value $c \in \mathbb{R}^k$ and restrict T to the level set $N_c = \mathcal{J}^{-1}(c) \subset M^2$. By Proposition 3.3, the group G acts on N_c . If \tilde{M} is a smooth manifold, then $\tilde{N} = N_c/G$ is also a smooth manifold, and T defines a map $\tilde{T}: \tilde{N} \rightarrow \tilde{N}$. It is symplectic with respect to the quotient symplectic structure $\tilde{\omega}$ on \tilde{N} . We would like to represent \tilde{T} as a discrete Lagrangian system with quotient configuration space \tilde{M} and Lagrangian \tilde{L} on $\tilde{M} \times \tilde{M}$. For this reduction we need

Non-degeneracy assumption. The matrix $G = (g^{\alpha\beta})$,

$$g^{\alpha\beta}(x, y) = \langle B(x, y) \mathbf{w}^\alpha(x), \mathbf{w}^\beta(y) \rangle, \quad x, y \in M, \quad (3.3)$$

is nondegenerate.

First let $c = 0$. Let $f(x, y) \in \mathbb{R}^k$ be the critical point of the function $s \mapsto L(x, \psi_s(y))$, provided it exists and is unique. Note that the Hessian of this function equals $G(x, \psi_s(y))$, and so is nondegenerate. The reduced Lagrangian is defined by

$$\tilde{L}(x, y) = L(x, \psi_{f(x, y)}(y)). \quad (3.4)$$

Since $\tilde{L}(x, y) = \tilde{L}(\tilde{x}, \tilde{y})$ depends only on $\tilde{x} = \pi(x)$ and $\tilde{y} = \pi(y)$, it is a function on $\widetilde{M} \times \widetilde{M}$.

Suppose now that $c = (c^1, \dots, c^k) \neq 0$. Locally there exist smooth functions ϕ_β , $\beta = 1, \dots, k$, on M such that $D_{\mathbf{w}^\alpha} \phi_\beta = \delta_\beta^\alpha$. In general there are topological obstructions to the existence of single valued globally defined ϕ_α . However, if $\pi: M \rightarrow \widetilde{M}$ has fibre \mathbb{R}^k , then ϕ_α exist globally. Since we work in a neighbourhood of a periodic orbit, this is irrelevant for us.

Replace L by gauge-equivalent Lagrangian⁵

$$\widehat{L}(x, y) = L(x, y) + c^\alpha (\phi_\alpha(y) - \phi_\alpha(x)).$$

Then \mathcal{J}^α is replaced by

$$\widehat{\mathcal{J}}^\alpha = \mathcal{J}^\alpha - c^\beta D_{\mathbf{w}^\alpha} \phi_\beta = \mathcal{J}^\alpha - c^\alpha.$$

Now $\hat{c} = 0$ and so \tilde{L} can be defined by (3.4) with L replaced by \widehat{L} .

Here is Routh' Theorem for discrete Lagrangian systems.

Proposition 3.4. *The projection $\pi: M \rightarrow \widetilde{M}$ takes trajectories of the system (M, L) with $\mathcal{J} = c$ to trajectories of the reduced Lagrangian system $(\widetilde{M}, \tilde{L})$.*

Next we give a coordinate version of the Routh reduction. Since the result is local, it is sufficient to perform the reduction near a given trajectory $\mathbf{x}^0 = (x_i^0)$. Since the vector fields \mathbf{w}^α are independent and commute, in a neighbourhood U_i of the point x_i^0 there are local coordinates $y_i \in \mathbb{R}^{m-k}$, $z_i \in \mathbb{R}^k$ such that $\mathbf{w}^\alpha|_{U_i} = \partial/\partial z_i^\alpha$. Similarly to the continuous case coordinates z_i^α are called cyclic. The variables $y_i \in \mathbb{R}^{m-k}$ are local coordinates on \widetilde{M} . Equation (3.2) means that

$$L(x_i, x_{i+1}) = \mathcal{L}(y_i, y_{i+1}, u_i), \quad u_i = z_{i+1} - z_i.$$

By (3.3), the matrix

$$G_i = -\left(\frac{\partial^2 L}{\partial z_i^\alpha \partial z_{i+1}^\beta} \right) = \left(\frac{\partial^2 \mathcal{L}}{\partial u_i^\alpha \partial u_i^\beta} \right) = (g_i^{\alpha\beta}) \quad (3.5)$$

⁵We use Einstein's sum rule with respect to repeated Greek indices, but not Latin indices.

is nondegenerate.

Without loss of generality we assume that $c = 0$. Then

$$\mathcal{J} = \partial_{u_i} \mathcal{L}(y_i, y_{i+1}, u_i) = 0. \quad (3.6)$$

Equation (3.6) can be locally solved with respect to $u_i = f_i(y_i, y_{i+1})$. Then the Routh function is defined by

$$\tilde{L}(y_i, y_{i+1}) = \mathcal{L}(y_i, y_{i+1}, f_i(y_i, y_{i+1})). \quad (3.7)$$

Hence

$$\partial_{y_i} \tilde{L}(y_i, y_{i+1}) = \partial_{y_i} \mathcal{L}(y_i, y_{i+1}, u_i) \Big|_{u_i=f_i(y_i, y_{i+1})}, \quad (3.8)$$

and similarly for the derivative with respect to y_{i+1} .

Suppose $x_i = (y_i, z_i)$ is a trajectory of the system (M, L) with $\mathcal{J} = 0$. Then

$$\partial_{y_i} (\mathcal{L}(y_{i-1}, y_i, u_{i-1}) + \mathcal{L}(y_i, y_{i+1}, u_i)) = 0.$$

By (3.8),

$$\partial_{y_i} (\tilde{L}(y_{i-1}, y_i) + \tilde{L}(y_i, y_{i+1})) = 0,$$

so (y_i) is a trajectory of the reduced system (\tilde{M}, \tilde{L}) .

To finish the proof of Routh's Theorem, it remains to show that \tilde{L} is a discrete Lagrangian, that is, it satisfies the twist condition. This follows from

Lemma 3.1. *Let $\tilde{B}_i = \tilde{B}(y_i, y_{i+1}) = -\partial_{12} \tilde{L}(y_i, y_{i+1})$ and $B_i = B(x_i, x_{i+1})$. Then*

$$\det \tilde{B}_i = \frac{\det B_i}{\det G_i} \neq 0. \quad (3.9)$$

Proof. Putting $(g_{\alpha\beta i}) = G_i^{-1}$ we have

$$\frac{\partial}{\partial y_i} f_\alpha(y_i, y_{i+1}) = -g_{\alpha\beta i} \frac{\partial^2}{\partial y_i \partial u_i^\beta} \mathcal{L}(y_i, y_{i+1}, u_i) \Big|_{u_i=f_i(y_i, y_{i+1})}. \quad (3.10)$$

We differentiate (3.8) with respect to y_{i+1} using (3.10):

$$-\tilde{B}_i = \frac{\partial^2 \mathcal{L}}{\partial y_i \partial y_{i+1}} + \frac{\partial f_\alpha}{\partial y_{i+1}} \frac{\partial^2 \mathcal{L}}{\partial u_i^\alpha \partial y_i} = \frac{\partial^2 L}{\partial y_i \partial y_{i+1}} + g_{\alpha\beta i} \frac{\partial^2 L}{\partial y_{i+1} \partial z_{i+1}^\alpha} \frac{\partial^2 L}{\partial y_i \partial z_i^\beta},$$

or in the matrix notation $\tilde{B}_i^{jk} = B_i^{jk} - B_i^{\alpha j} g_{\alpha\beta i} B_i^{k\beta}$. It remains to make an exercise in linear algebra. \square

3.4 Symplectic reduction for the Poincaré map

Suppose that the periodic trajectory is degenerate. Then the linear Poincaré map $P: W \rightarrow W$ has a unit eigenvalue: there exists $w \neq 0$ such that $Pw = w$. Since P is symplectic, $\omega(w, Pu) = \omega(w, u)$, and so $J_w(u) = \omega(w, u)$ is a linear first integral of P . Then it is possible to reduce P to a linear symplectic map $\tilde{P}: \tilde{W} \rightarrow \tilde{W}$ of lower dimension.

This section deals with symplectic linear algebra, and the origin of the symplectic map P is irrelevant. In particular, the notations below will be used both for discrete and continuous Lagrangian systems.

Suppose there are several eigenvectors corresponding to unit eigenvalue. Let $V \subset \{w \in W : Pw = w\}$. Then P has a first integral $J: W \rightarrow V^*$: for $w \in V$, $\langle J(u), w \rangle = J_w(u)$. We assume that V is isotropic: $\omega|_V = 0$. Then $V \subset J^{-1}(0)$. We put $\tilde{W} = J^{-1}(0)/V$.

Proposition 3.5 (Poincaré). *P generates a reduced symplectic operator $\tilde{P}: \tilde{W} \rightarrow \tilde{W}$ such that the diagram*

$$\begin{array}{ccc} V & \xrightarrow{P} & V \\ \downarrow & & \downarrow \\ \tilde{W} & \xrightarrow{\tilde{P}} & \tilde{W} \end{array}$$

is commutative. Furthermore,

$$\det(P - \rho I_W) = (1 - \rho)^{2k} \det(\tilde{P} - \rho I_{\tilde{W}}), \quad k = \dim V.$$

3.5 Routh reduction for linear discrete Lagrangian systems

Next we translate Proposition 3.5 to the language of the variational system, that is, the linear Lagrangian system (E, Λ) . To any eigenvector w of the Poincaré map P there corresponds a non-zero n -periodic solution $\mathbf{w} = (w_i)$ of the variational system. To the periodic solution \mathbf{w} there corresponds a linear periodic first integral

$$I_j(u_j, u_{j+1}) = \langle B_j w_j, u_{j+1} \rangle - \langle B_j u_j, w_{j+1} \rangle.$$

Indeed, if $\mathbf{u} = (u_j)$ is a solution of (2.14), then

$$\begin{aligned} 0 &= \langle A_j w_j - B_j^* w_{j+1} - B_{j-1} w_{j-1}, u_j \rangle - \langle A_j u_j - B_j^* u_{j+1} - B_{j-1} u_{j-1}, w_j \rangle \\ &= I_j(u_j, u_{j+1}) - I_{j-1}(u_{j-1}, u_j). \end{aligned}$$

Hence

$$I_{j-1}(u_{j-1}, u_j) = I_j(u_j, u_{j+1}). \quad (3.11)$$

In fact, $I_j(u_j, u_{j+1}) = J_w(u)$, where J_w is the integral of the Poincaré map and $u \in W$ corresponds to the trajectory (u_j) .

Suppose now there are several eigenvectors and let $V \subset \text{Ker}(P-I)$ be an isotropic subspace. Denote by $\Gamma \subset X$ the set of periodic trajectories corresponding to V . Let $w^\alpha \in V$, $\alpha = 1, \dots, k$, be a basis in V . Then the corresponding periodic trajectories $\mathbf{w}^\alpha = (w_i^\alpha)$ form a basis in γ . Let

$$I_j^\alpha(u_j, u_{j+1}) = \langle B_j w_j^\alpha, u_{j+1} \rangle - \langle B_j u_j, w_{j+1}^\alpha \rangle$$

be the corresponding integrals of the variational system. Since V is isotropic, the integrals commute:

$$I_j^\alpha(w_j^\beta, w_{j+1}^\beta) = \langle B_j w_j^\alpha, w_{j+1}^\beta \rangle - \langle B_j w_j^\beta, w_{j+1}^\alpha \rangle = \omega(w^\alpha, w^\beta) = 0. \quad (3.12)$$

We sometimes write $I_j = (I_j^1, \dots, I_j^k)$.

Below we need several non-degeneracy conditions.

Condition A. *The symmetric matrix*

$$G_i = (g_i^{\alpha\beta}), \quad g_i^{\alpha\beta} = \langle B_i w_i^\alpha, w_{i+1}^\beta \rangle,$$

is nondegenerate for all i . Denote $(g_{\alpha\beta i}) = (g_i^{\alpha\beta})^{-1} = G_i^{-1}$.

Condition B. *The matrix*

$$\overline{G} = \sum_{i=1}^n G_i^{-1} = (\bar{g}_{\alpha\beta}), \quad \bar{g}_{\alpha\beta} = \sum_{i=1}^n g_{\alpha\beta i}, \quad (3.13)$$

is nondegenerate.

Many of our results hold without condition B, so we impose it later. Condition A is used almost everywhere, so we impose it now. In the case of CLS, an analogue of condition A is also introduced, but finally it turns out to be unessential, see § A.3. An analogue of condition B is always satisfied for CLS.

Denote

$$F_i = \{w_i : \mathbf{w} \in \Gamma\} = \text{span}(w_i^1, \dots, w_i^k) \subset E_i.$$

The reduced Poincaré map \tilde{P} corresponds to the reduced linear Lagrangian system $(\tilde{E}, \tilde{\Lambda})$ with $\tilde{E}_i = E_i/F_i$ which is obtained by the Routh reduction of the system (E, Λ) . Under the non-degeneracy condition A, $\dim F_i = k$ and the reduced configuration space $\tilde{E}_i = E_i/F_i$ can be identified with

$$E_i^\perp = \{u \in E_i : \langle B_{i-1} w_{i-1}^\alpha, u \rangle = 0, \alpha = 1, \dots, k\}$$

via the projection $\Pi_i: E_i \rightarrow E_i^\perp$:

$$\Pi_i u = u_i - g_{\alpha\beta i} \langle B_{i-1} w_{i-1}^\alpha, u \rangle w_i^\beta. \quad (3.14)$$

We represent any vector $u_i \in E_i$ as

$$u_i = v_i + \lambda_{\beta i} w_i^\beta, \quad v_i = \Pi_i u_i \in E_i^\perp, \quad \lambda_{\beta i} = g_{\alpha\beta i-1} \langle B_{i-1} w_{i-1}^\alpha, u \rangle. \quad (3.15)$$

The Routh reduction for DLS is described by the following

Theorem 3.1. *Let $\mathbf{u} = (u_i)$, $u_i \in E_i$, be a trajectory of the system (E, Λ) such that $I_i(u_i, u_{i+1}) = 0$. Then $\mathbf{v} = (v_i)$, $v_i = \Pi_i u_i \in E_i^\perp$, is a trajectory of the linear Lagrangian system (E^\perp, Λ^\perp) with the Lagrangian*

$$\Lambda_i^\perp(v_i, v_{i+1}) = \frac{1}{2} \langle A_i v_i, v_i \rangle - \langle B_i v_i, v_{i+1} \rangle - \frac{1}{2} \langle C_i v_i, v_i \rangle,$$

where

$$\langle C_i v_i, v_i \rangle = g_{\alpha\beta i} \langle B_i v_i, w_{i+1}^\alpha \rangle \langle B_i v_i, w_{i+1}^\beta \rangle.$$

Conversely, if \mathbf{v} is a trajectory of the system (E^\perp, Λ^\perp) , then there exists a trajectory \mathbf{u} of the system (E, Λ) , defined mod Γ such that $I_i(u_i, u_{i+1}) = 0$ and $\Pi \mathbf{u} = \mathbf{v}$.

For the proof we will need an evident

Lemma 3.2. *Let \mathbf{u} , \mathbf{v} be such that $u_i - v_i \in F_i$:*

$$u_i = v_i + \lambda_{\beta i} w_i^\beta.$$

Then $I_i^\alpha(u_i, u_{i+1}) = c^\alpha$ for all $\alpha = 1, \dots, k$ and all i if and only if

$$\Delta \lambda_{\alpha i} = \lambda_{\alpha i+1} - \lambda_{\alpha i} = g_{\alpha\beta i} (c^\beta - I_i^\beta(v_i, v_{i+1})). \quad (3.16)$$

Equation (3.16) follows from

$$c^\alpha = I_i^\alpha(u_i, u_{i+1}) = I_i^\alpha(v_i, v_{i+1}) + g_i^{\alpha\beta} \Delta \lambda_{\beta i}.$$

Proof of Theorem 3.1. Let $\mathbf{u} = (u_i)$, $u_i \in E_i$, be a trajectory of (E, Λ) such that $I_i^\alpha(u_i, u_{i+1}) = 0$. Then for any variation $\phi_i \in E_i$ such that $\phi_i = 0$ except for $i = 1, \dots, n$, we have

$$0 = h(\mathbf{u}, \phi) = \sum_{i=1}^n \langle A_i u_i - B_i^* u_{i+1} - B_{i-1} u_{i-1}, \phi_i \rangle.$$

Let $\mathbf{v} = \Pi \mathbf{u}$. By (3.15) and (3.16),

$$I_i^\alpha(v_i, v_{i+1}) = -\langle B_i v_i, w_{i+1}^\alpha \rangle, \quad \Delta \lambda_{\alpha i} = g_{\alpha \beta i} \langle B_i v_i, w_{i+1}^\beta \rangle.$$

Choose the variation ϕ such that $\phi_i \in E_i^\perp$. Using

$$A_i w_i^\alpha = B_{i-1} w_{i-1}^\alpha + B_i^* w_{i+1}^\alpha \quad (3.17)$$

and $\langle B_{i-1} w_{i-1}^\beta, \phi_i \rangle = 0$, we obtain

$$\begin{aligned} h(\mathbf{u}, \phi) &= \sum_{i=1}^n \langle A_i v_i - B_i^* v_{i+1} - B_{i-1} v_{i-1}, \phi_i \rangle - \sum_{i=1}^n \Delta \lambda_{\beta i} \langle B_i \phi_i, w_{i+1}^\beta \rangle \\ &= h(\mathbf{v}, \phi) - \sum_{i=1}^n g_{\alpha \beta i} \langle B_i v_i, w_{i+1}^\alpha \rangle \langle B_i \phi_i, w_{i+1}^\beta \rangle = h^\perp(\mathbf{v}, \phi), \end{aligned}$$

where

$$h^\perp(\mathbf{v}, \mathbf{v}) = h(\mathbf{v}, \mathbf{v}) - \sum_{i=1}^n g_{\alpha \beta i} \langle B_i v_i, w_{i+1}^\alpha \rangle \langle B_i v_i, w_{i+1}^\beta \rangle. \quad (3.18)$$

This is the quadratic action functional for the system (E^\perp, Λ^\perp) . Since $\phi_i \in E_i^\perp$, $i = 1, \dots, n$, are arbitrary, \mathbf{v} is a trajectory of (E^\perp, Λ^\perp) . We skip the proof of the converse. \square

For n -periodic \mathbf{v} the bilinear form h^\perp on $X^\perp = E_1^\perp \times \dots \times E_n^\perp$ equals

$$h^\perp(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^n \langle A_i^\perp v_i - B_i^{\perp *} v_{i+1} - B_{i-1}^\perp v_{i-1}, v_i \rangle, \quad v_{n+1} = v_1, \quad v_0 = v_n, \quad (3.19)$$

where

$$A_i^\perp = R_i(A_i - C_i), \quad B_i^\perp = R_{i+1} B_i$$

and $R_i: E_i^* \rightarrow (E_i^\perp)^*$ is the restriction map. The reduced variational system is

$$A_i^\perp v_i - B_i^{\perp *} v_{i+1} - B_{i-1}^\perp v_{i-1} = 0.$$

Let $\mathbf{H}^\perp, \mathbf{B}^\perp: X^\perp \rightarrow X^{\perp *}$ be the corresponding linear operators. We also put

$$\sigma^\perp \beta^\perp = \left(\prod \det B_i^\perp \right)^{-1}, \quad \beta^\perp > 0, \quad \sigma^\perp \in \{1, -1\}.$$

Then analogously to (2.12),

$$\det \mathbf{B}^\perp = (-1)^{m-k} \prod \det B_i^\perp.$$

Hill's theorem 2.1, applied to the reduced system, gives

Corollary 3.1. *The following reduced Hill formula holds:*

$$\det(\tilde{P} - I) = \frac{\det \mathbf{H}^\perp}{\det \mathbf{B}^\perp} = \sigma^\perp (-1)^{m-k} \beta^\perp \det \mathbf{H}^\perp.$$

To use this formula for stability problems, we need to know σ^\perp and the Morse index of h^\perp . However, the relation between the Morse indices of h and h^\perp is not evident. The reason is that a periodic sequence $\mathbf{v} \in X^\perp$ in general corresponds to a non-periodic sequence $\mathbf{u} \notin X$ such that $I_i(u_i, u_{i+1}) = 0$. We discuss this problem in the next two sections.

Lemma 3.1 implies $\text{ind } B_i = \text{ind } B_i^\perp + \text{ind } G_i \pmod{2}$. Therefore,

$$\sigma^\perp = (-1)^{\sum \text{ind } B_i^\perp} = \sigma (-1)^{\sum \text{ind } G_i}. \quad (3.20)$$

3.6 (

Degeneracy of h We denote by $\Gamma \subset X$ the space of periodic solutions corresponding to $V \subset \text{Ker}(P - I)$. It is spanned by $\mathbf{w}_1, \dots, \mathbf{w}_k \in X$. Since $\mathbf{H}\mathbf{w} = 0$ for $\mathbf{w} \in \Gamma$, the Hessian bilinear form $h(\mathbf{u}, \mathbf{u}) = \langle \mathbf{H}\mathbf{u}, \mathbf{u} \rangle$ is degenerate and defines a bilinear form \hat{h} on $\hat{X} = X/\Gamma$. To compare h with h^\perp we need to restrict h to the level set of I . Let

$$Y = \{\mathbf{u} \in X : I_1(u_1, u_2) = \dots = I_n(u_n, u_1)\}, \quad (3.21)$$

$$Z = \{\mathbf{v} \in X : v_i \in F_i\} = \{\mathbf{v} \in X : v_i = \lambda_{\alpha i} w_i^\alpha\}. \quad (3.22)$$

Proposition 3.6. *The spaces Y and Z are h -orthogonal, that is, $h(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u} \in Y$ and $\mathbf{v} \in Z$. Moreover, Y is the h -orthogonal complement to Z :*

$$Y = \{\mathbf{u} \in X : h(\mathbf{u}, \mathbf{v}) = 0 \text{ for all } \mathbf{v} \in Z\}.$$

The restriction of h to Z is

$$h(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^n g_i^{\alpha\beta} \Delta \lambda_{\alpha i} \Delta \lambda_{\beta i} = \sum_{i=1}^n \langle G_i \Delta \lambda_i, \Delta \lambda_i \rangle, \quad v_i = \lambda_{\alpha i} w_i^\alpha.$$

Proof. Let $\mathbf{u} \in Y$ and $\mathbf{v} = (\lambda_{\alpha i} \mathbf{w}_i^\alpha) \in Z$. Then

$$\begin{aligned}
h(\mathbf{u}, \mathbf{v}) &= \sum_{i=1}^n \langle A_i u_i - B_i^* u_{i+1} - B_{i-1} u_{i-1}, \lambda_{\alpha i} w_i^\alpha \rangle \\
&= \sum_{i=1}^n \lambda_{\alpha i} (\langle A_i u_i, w_i^\alpha \rangle - \langle B_i w_i^\alpha, u_{i+1} \rangle - \langle B_{i-1} u_{i-1}, w_i^\alpha \rangle) \\
&= \sum_{i=1}^n \lambda_{\alpha i} (\langle A_i w_i^\alpha, u_i \rangle - \langle B_i u_i, w_{i+1}^\alpha \rangle - \langle B_{i-1} w_{i-1}^\alpha, u_i \rangle) \\
&= \sum_{i=1}^n \langle A_i w_i^\alpha - B_i^* w_{i+1}^\alpha - B_{i-1} w_{i-1}^\alpha, \lambda_{\alpha i} u_i \rangle = 0
\end{aligned}$$

by (3.17); we used that

$$\langle B_i w_i^\alpha, u_{i+1} \rangle = \langle B_i u_i, w_{i+1}^\alpha \rangle + c^\alpha, \quad \langle B_{i-1} u_{i-1}, w_i^\alpha \rangle = \langle B_{i-1} w_{i-1}^\alpha, u_i \rangle - c^\alpha.$$

Conversely, if $\mathbf{v} = (\lambda_{\alpha i} w_i^\alpha)$ and $h(\mathbf{u}, \mathbf{v}) = 0$ for all λ_α , then

$$\langle A_i u_i - B_i^* u_{i+1} - B_{i-1} u_{i-1}, w_i^\alpha \rangle = 0.$$

Using (3.17) we obtain

$$I_i^\alpha(u_i, u_{i+1}) = I_{i-1}^\alpha(u_{i-1}, u_i).$$

Thus, $\mathbf{u} \in Y$.

Next we compute the restriction of h to Z . Let $\mathbf{v} = (\lambda_{\alpha i} \mathbf{w}_i^\alpha)$. Then by (3.17),

$$\begin{aligned}
h(\mathbf{v}, \mathbf{v}) &= \sum_{i=1}^n (\lambda_{\alpha i} \lambda_{\beta i} \langle A_i w_i^\alpha, w_i^\beta \rangle - \lambda_{\alpha i} \lambda_{\beta i-1} \langle B_{i-1} w_{i-1}^\alpha, w_i^\beta \rangle \\
&\quad - \lambda_{\alpha i} \lambda_{\beta i+1} \langle B_i w_i^\alpha, w_{i+1}^\beta \rangle) \\
&= \sum_{i=1}^n (\lambda_{\alpha i} (\lambda_{\beta i} - \lambda_{\beta i+1}) \langle B_i w_i^\alpha, w_{i+1}^\beta \rangle \\
&\quad + \lambda_{\beta i} (\lambda_{\alpha i} - \lambda_{\alpha i-1}) \langle B_{i-1} w_{i-1}^\beta, w_i^\alpha \rangle) \\
&= \sum_{i=1}^n (g_i^{\alpha\beta} \lambda_{\alpha i} (\lambda_{\beta i} - \lambda_{\beta i+1}) + g_{i-1}^{\alpha\beta} \lambda_{\alpha i} (\lambda_{\beta i} - \lambda_{\beta i-1})) \\
&= \sum_{i=1}^n g_i^{\alpha\beta} \Delta \lambda_{\alpha i} \Delta \lambda_{\beta i}.
\end{aligned}$$

□

We obtain a quadratic form on Z :

$$h|_Z(\mathbf{v}, \mathbf{v}) = \langle \mathbf{H}_Z \mathbf{v}, \mathbf{v} \rangle = \sum_{i=1}^n \langle G_i \Delta \lambda_i, \Delta \lambda_i \rangle = \langle \mathbf{G} \lambda, \lambda \rangle,$$

where $\lambda_i \in \mathbb{R}^k$, $\mathbf{H}_Z: Z \rightarrow Z^*$, and the operator $\mathbf{G}: \mathbb{R}^{kn} \rightarrow \mathbb{R}^{kn}$ is defined by

$$(\mathbf{G} \lambda)_i = G_{i-1} \Delta \lambda_{i-1} - G_i \Delta \lambda_i.$$

We have

$$\text{Ker } \mathbf{G} = \{ \lambda \in \mathbb{R}^{kn} : G_1 \Delta \lambda_1 = \dots = G_n \Delta \lambda_n \}.$$

Thus, $\Delta \lambda_i = G_i^{-1} c$. For $\lambda \in \text{Ker } \mathbf{G}$ the equation $\sum_{i=1}^n \Delta \lambda_i = 0$ implies $\overline{G} c = 0$, where \overline{G} is the matrix (3.13).

Now we impose the non-degeneracy assumption B: the matrix \overline{G} is nondegenerate. Then

$$\text{Ker } \mathbf{G} = \{ \lambda \in \mathbb{R}^{nk} : \lambda_1 = \dots = \lambda_n \} \quad \text{and} \quad \text{Ker } \mathbf{H}_Z = \Gamma.$$

Let $\widehat{Z} = Z/\Gamma$.

Proposition 3.7. *The form $h|_{\widehat{Z}}$ is nondegenerate and*

$$\text{ind } h|_{\widehat{Z}} = \sum_{i=1}^n \text{ind } G_i - \text{ind } \overline{G}. \quad (3.23)$$

Proof. By Proposition 3.6, in the coordinates $\mu_i = \Delta \lambda_i \in \mathbb{R}^k$ we have $h|_{\widehat{Z}} = \chi|_{\Theta}$, where χ is the following quadratic form on \mathbb{R}^{nk} :

$$\chi(\mu, \mu) = \sum_{i=1}^n \langle G_i \mu_i, \mu_i \rangle,$$

and the space $\Theta \subset \mathbb{R}^{nk}$ is defined by the condition

$$\Theta = \left\{ \mu \in \mathbb{R}^{nk} : \sum_{i=1}^n \mu_i = 0 \right\}.$$

Below we use the same notation χ for the corresponding bilinear form. Consider the k -dimensional space

$$\Xi = \{ \mu \in \mathbb{R}^{nk} : \mu_i = G_i^{-1} \nu, \nu \in \mathbb{R}^k, j = 1, \dots, n \}.$$

Since \overline{G} is nondegenerate, $\mathbb{R}^{nk} = \Xi \oplus \Theta$ and moreover, $\chi(\mu, \xi) = 0$ for any $\mu \in \Theta$ and $\xi \in \Xi$. Therefore, the spaces Ξ and Θ are χ -orthogonal and

$$\sum_{i=1}^n \text{ind } G_i = \text{ind } \chi = \text{ind } \chi|_{\Xi} + \text{ind } \chi|_{\Theta} = \text{ind } \overline{G} + \text{ind } h|_{\widehat{Z}}.$$

□

Proposition 3.8. $Y + Z = X$ and $Y \cap Z = \Gamma$.

This follows from

Lemma 3.3. *For any $\mathbf{v} \in X$ there exists $\mathbf{u} = \Phi \mathbf{v} \in Y$, unique mod Γ , such that $\mathbf{u} - \mathbf{v} \in Z$. Explicitly, $u_i = v_i + \lambda_{\beta_i} w_i^{\beta}$, where the λ_{β_i} satisfy (3.16) and*

$$c^{\alpha} = \kappa^{\alpha\beta} \sum_{i=1}^n g_{\beta\delta i} I_i^{\delta}(v_i, v_{i+1}), \quad (\kappa^{\alpha\beta}) = (\bar{g}_{\alpha\beta})^{-1} = \overline{G}^{-1}. \quad (3.24)$$

The map $\Phi: X \rightarrow \widehat{Y} = Y/\Gamma$ satisfies $\Phi(Z) = 0$ and $\Phi|_Y$ is the identity mod Γ . Proposition 3.8 follows immediately.

To prove Lemma 3.3, for given $\mathbf{v} \in X$ we find \mathbf{u} such that

$$I_i^{\alpha}(u_i, u_{i+1}) = c^{\alpha} = \text{const.}$$

By Lemma 3.2, the $\lambda_{\alpha i}$ satisfy (3.16). If $\lambda_{\alpha i}$ is n -periodic in i , then $\sum_{i=1}^n \Delta \lambda_{\alpha i} = 0$, which gives (3.24). If (3.24) holds, then equation (3.16) determines n -periodic $\lambda_{\alpha i}$ modulo a constant independent of i . Thus, \mathbf{u} is defined uniquely modulo Γ .

Let $\widehat{X} = X/\Gamma$ and let $\pi_{\Gamma}: X \rightarrow \widehat{X}$ be the corresponding canonical projection. Then there exists a linear map $\widehat{\Pi}: \widehat{X} \rightarrow X^{\perp}$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{\pi_{\Gamma}} & \widehat{X} \\ & \searrow \Pi & \swarrow \widehat{\Pi} \\ & X^{\perp} & \end{array}$$

Corollary 3.2. *The maps $\widehat{\Pi}|_{\widehat{Y}}: \widehat{Y} \rightarrow X^{\perp}$ and $\Phi|_{X^{\perp}}: X^{\perp} \rightarrow \widehat{Y}$ are mutually inverse isomorphisms.*

Proof. If $\Pi \mathbf{v} = 0$ for $\mathbf{v} \in Y$, then $\mathbf{v} \in Z$ and hence $\mathbf{v} \in Y \cap Z = \Gamma$ by Proposition 3.7. The equation $Y + Z = X$ implies that $\Pi|_Y$ is surjective. We also have $\Phi(Z) = 0$. □

The spaces $\widehat{Y}, \widehat{Z} \subset \widehat{X}$ are orthogonal with respect to the bilinear form \widehat{h} on \widehat{X} . By Proposition 3.7, \widehat{h} is nondegenerate on \widehat{Z} and its index is given by (3.23).

From the point of view of the Routh reduction it is natural to consider the space

$$Y^0 = \{\mathbf{u} \in X : I_i(u_i, u_{i+1}) = 0, i = 1, \dots, n\} \subset Y.$$

Define $\mathbf{d}_\alpha = (d_{\alpha i}) \in X^*$ by

$$d_{\alpha i} = g_{\alpha\beta i-1} B_{i-1} w_{i-1}^\beta - g_{\alpha\beta i} B_i^* w_{i+1}^\beta. \quad (3.25)$$

Note that $\langle d_{\alpha i}, w_i^\alpha \rangle = 0$. We have

$$\langle \mathbf{d}_\alpha, \mathbf{u} \rangle = \sum_{i=1}^n g_{\alpha\beta i} I_i^\beta(u_i, u_{i+1}), \quad \mathbf{u} \in X.$$

Proposition 3.9.

$$Y^0 + Z = \{\mathbf{v} \in X : \langle \mathbf{d}_\alpha, \mathbf{v} \rangle = 0, \alpha = 1, \dots, k\}. \quad (3.26)$$

Proof. Let $\mathbf{v} \in X$. Then $\mathbf{v} \in Y^0 + Z$, $\mathbf{u} = \Phi(\mathbf{v}) \in Y^0$. By Lemma 3.3,

$$0 = \sum_{i=1}^n g_{\alpha\beta i} I_i^\beta(v_i, v_{i+1}) = \sum_{i=1}^n \langle d_{\alpha i}, v_i \rangle = \langle \mathbf{d}_\alpha, \mathbf{v} \rangle.$$

□

Next we compute the restriction $h|_Y$. Let h^\perp be the bilinear action form for the reduced system (E^\perp, Λ^\perp) , and let $h^\top = h \circ \Phi|_{X^\perp}$, where Φ is the operator from Corollary 3.2, be the bilinear form on X^\perp corresponding to $h|_Y$, see the commutative diagram

$$\begin{array}{ccc} \widehat{Y} & & \\ \widehat{\Pi}|_{\widehat{Y}} \updownarrow \Phi & \nearrow \widehat{h}|_{\widehat{Y}} & \\ X^\perp & \searrow h^\top & \mathbb{R} \end{array}$$

Proposition 3.10. For any $\mathbf{v} \in X^\perp$,

$$h^\top(\mathbf{v}, \mathbf{v}) = h^\perp(\mathbf{v}, \mathbf{v}) + \bar{g}_{\alpha\beta} c^\alpha c^\beta, \quad (3.27)$$

where the coefficients $c^\alpha(\mathbf{v})$ are defined by (3.24).

This follows from a more general formula which we prove next.

Lemma 3.4. *Let $\mathbf{v} \in X$ and $\mathbf{u} = \Phi(\mathbf{v})$. Then*

$$h(\mathbf{u}, \mathbf{u}) = h(\mathbf{v}, \mathbf{v}) - \bar{g}_{\alpha\beta} c^\alpha c^\beta + \sum_{i=1}^n g_{\alpha\beta i} I_i^\alpha(v_i, v_{i+1}) I_i^\beta(v_i, v_{i+1}), \quad (3.28)$$

where the coefficients $c^\alpha(\mathbf{v})$ are defined by (3.24).

Proof. By Proposition 3.6,

$$h(\mathbf{u}, \mathbf{u}) = h(\mathbf{v}, \mathbf{v}) + \sum_{i=1}^n g_i^{\alpha\beta} \Delta \lambda_{\alpha i} \Delta \lambda_{\beta i}.$$

By (3.16) and (3.24),

$$\begin{aligned} \sum_{i=1}^n g_i^{\alpha\beta} \Delta \lambda_{\alpha i} \Delta \lambda_{\beta i} &= \sum_{i=1}^n g_{\alpha\beta i} (c^\alpha - I_i^\alpha(v_i, v_{i+1})) (c^\beta - I_i^\beta(v_i, v_{i+1})) \\ &= \sum_{i=1}^n g_{\alpha\beta i} I_i^\alpha(v_i, v_{i+1}) I_i^\beta(v_i, v_{i+1}) - \bar{g}_{\alpha\beta} c^\alpha c^\beta. \end{aligned}$$

□

Proof of Proposition 3.10. We use (3.28) with $\mathbf{v} \in X^\perp$. Then

$$I_i^\alpha(v_i, v_{i+1}) = -\langle B_i v_i, w_{i+1}^\alpha \rangle,$$

and so

$$\sum_{i=1}^n g_{\alpha\beta i} I_i^\alpha(v_i, v_{i+1}) I_i^\beta(v_i, v_{i+1}) = \sum_{i=1}^n g_{\alpha\beta i} \langle B_i v_i, w_{i+1}^\alpha \rangle \langle B_i v_i, w_{i+1}^\beta \rangle.$$

Last we use (3.18). □

3.7 The indices of h and h^\perp

In this subsection we discuss the relation between $\text{ind } h|_Y = \text{ind } \hat{h}|_{\hat{Y}}$ and $\text{ind } h^\perp$. Using the isomorphisms $\Phi: X^\perp \rightarrow \hat{Y}$ and $\hat{\Pi}: \hat{Y} \rightarrow X^\perp$, we compare instead the indices of h^\perp and $h^\top = h \circ \Phi$ on X^\perp . As mentioned earlier, h^\top and h^\perp coincide on $X_0^\perp = \Pi Y^0$.

We need some assumptions on the unit eigenspace of the linear Poincaré map $P: W \rightarrow W$. Suppose $V = \text{Ker}(P - I)$ is isotropic. It is well known (see, for example,

[27]) that the generalized eigenspace $N = \text{Ker}(P - I)^{2m}$ is symplectic. Since $V \subset N$ is isotropic, $\dim N \geq 2k$. We consider the least degenerate case $\dim N = 2k$. Then

$$N = \text{Ker}(P - I)^2 = \{v \in W : Pv - v \in V\}$$

is symplectic and $V = \text{Ker}(P - I)$ is a Lagrangian subspace of N . Consider the bilinear form

$$b(v, w) = \omega((P - I)v, w), \quad v, w \in N. \quad (3.29)$$

A computation shows [27] that b is symmetric.

Let Σ be the set of trajectories \mathbf{v} of the variational system corresponding to the vectors $v \in N$. Then $\Gamma \subset \Sigma$. Trajectories in $\Sigma \setminus \Gamma$ are not periodic. The projection

$$\Omega^\perp = \Pi\Sigma \subset X^\perp$$

consists of periodic sequences. We have a natural map $\Psi = \Phi\Pi: \Sigma \rightarrow \hat{Y}$. Set $\hat{\Omega} = \Psi\Sigma$. We will see that Ω^\perp is orthogonal to $X_0^\perp = \Pi Y^0$ with respect to h^\perp , and $\hat{\Omega}$ is orthogonal to \hat{Y} with respect to h . Since h^\top and h^\perp coincide on X_0^\perp , the difference of their indices is determined by their restrictions to the complement of X_0^\perp . Thus if $\Omega^\perp \oplus X_0^\perp = X^\perp$, the difference of the indices is determined by $h^\top|_{\Omega^\perp}$ and $h^\perp|_{\Omega^\perp}$. However, in general $X^\perp \neq \Omega^\perp \oplus X_0^\perp$. To ensure this expansion we need an extra non-degeneracy condition C below.

Take a basis w^1, \dots, w^k in V and the conjugate basis q_1, \dots, q_k in a Lagrangian complement of V in N . Then $w^1, \dots, w^k, q_1, \dots, q_k$ is a symplectic basis in N and

$$\omega(w^\alpha, w^\beta) = 0, \quad \omega(q_\alpha, q_\beta) = 0, \quad \omega(w^\alpha, q_\beta) = \delta_\beta^\alpha, \quad Pq_\alpha = q_\alpha + s_{\alpha\beta}w^\beta. \quad (3.30)$$

Combining (3.29) and (3.30), we obtain

$$b(q_\alpha, q_\beta) = s_{\alpha\beta}.$$

Hence the matrix $S = (s_{\alpha\beta})$ is symmetric. Define symmetric matrices $A = (a_{\alpha\beta})$ and $A^\perp = (a_{\alpha\beta}^\perp)$:

$$a_{\alpha\beta} = s_{\alpha\delta}\kappa^{\delta\varepsilon}s_{\varepsilon\beta} - s_{\alpha\beta}, \quad a_{\alpha\beta}^\perp = s_{\alpha\beta} - \bar{g}_{\alpha\beta}. \quad (3.31)$$

Below we need another non-degeneracy assumption.

Condition C. *The matrix $A^\perp = (a_{\alpha\beta}^\perp)$ is nondegenerate.*

Definition 3.1. *We say that a periodic trajectory is nondegenerate mod V if the non-degeneracy conditions A, B, and C hold.*

Theorem 3.2. *Suppose that \mathbf{x} is nondegenerate mod V . Then*

$$\text{ind } h - \text{ind } h|_Z - \text{ind } h^\perp = \text{ind } A - \text{ind } A^\perp. \quad (3.32)$$

We prove Theorem 3.2 in §3.8.

Corollary 3.3. *Suppose that \mathbf{x} is nondegenerate mod V . Then*

$$\text{ind } h = \text{ind } h^\perp + \text{ind } h|_Z + \text{ind } b + \text{ind } \overline{G} \pmod{2}.$$

Indeed, $SA^\perp = A\overline{G}$ implies

$$\text{sign det } A = \text{sign}(\text{det } S \text{ det } G \text{ det } A^\perp).$$

Therefore, $\text{ind } A - \text{ind } A^\perp = \text{ind } b + \text{ind } \overline{G} \pmod{2}$.

Proposition 3.7 implies

$$\text{Corollary 3.4. } \text{ind } h = \text{ind } h^\perp + \sum_{i=1}^n \text{ind } G_i + \text{ind } b \pmod{2}.$$

Equation (3.20) combined with Corollary 3.4 imply

$$\text{Corollary 3.5. } \sigma(-1)^{\text{ind } h} = \sigma^\perp(-1)^{\text{ind } h^\perp + \text{ind } b}.$$

3.8 The spaces Ω and Ω^\perp

Consider solutions

$$\mathbf{w}^\alpha = (w_i^\alpha), \quad \mathbf{q}_\alpha = (q_{\alpha i})$$

of the variational system corresponding to the symplectic basis w^α , q_α of the space N . They form a basis in Σ . The solutions $\mathbf{w}^\alpha \in \Gamma$ are n -periodic and satisfy (3.12). Equation (3.30) implies that \mathbf{q}_α satisfy

$$q_{\alpha, i+n} - q_{\alpha i} = s_{\alpha\beta} w_i^\beta. \quad (3.33)$$

Since the basis is symplectic,

$$I_i^\alpha(q_{\beta i}, q_{\beta i+1}) = \langle B_i w_i^\alpha, q_{\beta i+1} \rangle - \langle B_i q_{\beta i}, w_{i+1}^\alpha \rangle = \omega(w^\alpha, q_\beta) = \delta_\beta^\alpha, \quad (3.34)$$

$$\langle B_j q_{\alpha j}, q_{\beta j+1} \rangle - \langle B_j q_{\beta j}, q_{\alpha j+1} \rangle = \omega(q_\alpha, q_\beta) = 0. \quad (3.35)$$

Let $\mathbf{q}_\alpha^\perp = \Pi \mathbf{q}_\alpha \in \Omega^\perp \subset X^\perp$. Then $q_{\alpha i}^\perp = q_{\alpha i} - \lambda_{\alpha\beta i}^\perp w_i^\beta$, where

$$\lambda_{\alpha\beta i+n}^\perp - \lambda_{\alpha\beta i}^\perp = s_{\alpha\beta}, \quad \lambda_{\alpha\beta i}^\perp = g_{\gamma\beta i-1} \langle B_{i-1} w_{i-1}^\gamma, q_{\alpha i} \rangle. \quad (3.36)$$

For any $\alpha = 1, \dots, k$ define $\widehat{\mathbf{q}}_\alpha \in \widehat{Y}$ by $\widehat{\mathbf{q}}_\alpha = \Psi \mathbf{q}_\alpha$. Then

$$\hat{q}_{\alpha i} = q_{\alpha i} - \nu_{\alpha \beta i} w_i^\beta,$$

where the coefficients $\nu_{\alpha \beta i}$ are chosen so that the \hat{q}_α are periodic and

$$I_j(\hat{q}_{\beta j}, \hat{q}_{\beta j+1}) = -\langle B_j \hat{q}_{\beta j}, w_{j+1}^\alpha \rangle + \langle B_j w_j^\alpha, \hat{q}_{\beta j+1} \rangle = c_\beta^\alpha \quad (3.37)$$

are independent of j . Then

$$\nu_{\alpha \beta i+n} - \nu_{\alpha \beta i} = s_{\alpha \beta}, \quad \Delta \nu_{\alpha \beta i} = s_{\alpha \gamma} \kappa^{\gamma \delta} g_{\delta \beta i}, \quad c_\beta^\alpha = \delta_\beta^\alpha - s_{\beta \gamma} \kappa^{\gamma \alpha}. \quad (3.38)$$

Since $\widehat{\mathbf{q}}_\alpha$ is defined modulo Γ , we have $\widehat{\mathbf{q}}_\alpha \in \widehat{Y}$ and $\widehat{\mathbf{q}}_\alpha = \Phi \mathbf{q}_\alpha^\perp$.

We have the spaces

$$\Omega^\perp = \Pi \Sigma = \text{span}(\mathbf{q}_1^\perp, \dots, \mathbf{q}_k^\perp) \subset X^\perp, \quad \widehat{\Omega} = \Psi \Sigma = \Phi \Omega^\perp = \text{span}(\widehat{\mathbf{q}}_1, \dots, \widehat{\mathbf{q}}_k) \subset \widehat{Y}.$$

We define \widehat{Y}^0 as the image of Y^0 under the projection $\pi_\Gamma: X \rightarrow \widehat{X} = X/\Gamma$.

Theorem 3.3. *Suppose that the non-degeneracy conditions A, B, and C hold. Then*

- (a) $\widehat{Y} = \widehat{\Omega} \oplus \widehat{Y}^0$ and $X^\perp = \Omega^\perp \oplus X_0^\perp$;
- (b) the maps $\widehat{\Pi}|_{\widehat{\Omega}}: \widehat{\Omega} \rightarrow \Omega^\perp$ and $\widehat{\Pi}|_{\widehat{Y}^0}: \widehat{Y}^0 \rightarrow X_0^\perp$ are linear isomorphisms;
- (c) $h|_{Y^0} = h^\perp|_{X_0^\perp} \circ \Pi|_{Y^0}$, $\hat{h}|_{\widehat{Y}^0} = h^\perp|_{X_0^\perp} \circ \widehat{\Pi}|_{\widehat{Y}^0}$;
- (d) for any $\mathbf{u} \in Y^0$ and $\alpha, \beta = 1, \dots, k$,

$$h(\widehat{\mathbf{q}}_\alpha, \mathbf{u}) = 0, \quad h(\widehat{\mathbf{q}}_\alpha, \widehat{\mathbf{q}}_\beta) = a_{\alpha \beta};$$

- (e) for any $\mathbf{u}^\perp \in \Pi Y^0$ and $\alpha, \beta = 1, \dots, k$,

$$h^\perp(\mathbf{q}_\alpha^\perp, \mathbf{u}^\perp) = 0, \quad h^\perp(\mathbf{q}_\alpha^\perp, \mathbf{q}_\beta^\perp) = a_{\alpha \beta}^\perp.$$

The proof of Theorem 3.3 is contained in § A.1.

Now we prove Theorem 3.2. Recall that by Proposition 3.6, $Y \perp_h Z$ and $\widehat{Y} \perp_{\hat{h}} \widehat{Z}$. By Theorem 3.3 we have the \hat{h} -orthogonal expansion $\widehat{X} = \widehat{Z} \oplus \widehat{\Omega} \oplus \widehat{Y}^0$ and the h^\perp -orthogonal expansion $X^\perp = \Omega^\perp \oplus X_0^\perp$ and $\hat{h}|_{\widehat{Y}^0} = h^\perp|_{\Pi \widehat{Y}^0} \circ \widehat{\Pi}|_{\widehat{Y}^0}$. Therefore,

$$\text{ind } \hat{h} - \text{ind } \hat{h}|_{\widehat{Z}} - \text{ind } h^\perp = \text{ind } \hat{h}|_{\widehat{\Omega}} - \text{ind } h^\perp|_{\Omega^\perp} = \text{ind}(a_{\alpha \beta}) - \text{ind}(a_{\alpha \beta}^\perp).$$

It remains to use the equations

$$\text{ind } h = \text{ind } \hat{h}, \quad \text{ind } h|_Z = \text{ind } \hat{h}|_{\widehat{Z}}, \quad \text{ind } h|_\Omega = \text{ind } \hat{h}|_{\widehat{\Omega}}.$$

3.9 Degeneracy for ρ -index form

The connection between the indices for the original and the reduced systems is much simpler for the ρ -index form. We take complex $\rho \in S^1$ and perform the same computation for the corresponding Hermitian form h_ρ on the complex space

$$X_\rho = \{\mathbf{u} = (u_j)_{j \in \mathbb{Z}} : u_j \in E_j, u_{j+n} = \rho u_j\}$$

of quasiperiodic sequences. We define $Y_\rho, Z_\rho \subset X_\rho$ by the same formulae (3.21):

$$Y_\rho = \{\mathbf{u} \in X_\rho : I^1(u_1, u_2) = \cdots = I^n(u_n, u_{n+1})\}, \quad (3.39)$$

$$Z_\rho = \{\mathbf{v} \in X_\rho : v_i = \lambda_{\alpha i} w_j^\alpha, \lambda_{\alpha i+n} = \rho \lambda_{\alpha j}\}. \quad (3.40)$$

The main difference is that for $\rho \neq 1$, $\mathbf{w}^\alpha \notin X_\rho$. This implies, in particular, that

$$Y_\rho = Y_\rho^0 = \{\mathbf{u} \in X_\rho : I_j(u_j, u_{j+1}) = 0, j = 1, \dots, n\},$$

Proposition 3.11. *The spaces Y_ρ, Z_ρ are h -orthogonal, that is, $h(\mathbf{u}, \bar{\mathbf{v}}) = 0$ for all $\mathbf{u} \in Y_\rho$ and $\mathbf{v} \in Z_\rho$. If $h(\mathbf{u}, \bar{\mathbf{v}}) = 0$ for all $\mathbf{v} \in Z_\rho$, then $\mathbf{u} \in Y_\rho$. The restriction of h_ρ to Z_ρ is given by*

$$h(\mathbf{v}, \bar{\mathbf{v}}) = \sum_{i=1}^n g_i^{\alpha\beta} \Delta \lambda_{\alpha i} \Delta \bar{\lambda}_{\beta i} = \sum_{i=1}^n \langle G_i \Delta \lambda_i, \Delta \bar{\lambda}_i \rangle, \\ \mathbf{v} = (\lambda_{\alpha i} \mathbf{w}_i^\alpha), \quad \lambda_{\alpha n+i} = \rho \lambda_{\alpha i}.$$

The proof is the same as for $\rho = 1$ (see Proposition 3.6).

Suppose assumption A holds.⁶ Then for $\rho \neq 1$, Y_ρ is the h -orthogonal complement to Z_ρ and $h|_{Z_\rho}$ is nondegenerate. Indeed,

$$h|_Z(\mathbf{v}, \mathbf{v}) = \langle \mathbf{G}_\rho \lambda, \bar{\lambda} \rangle,$$

where $\lambda_i \in \mathbb{C}^k$ and

$$(\mathbf{G}_\rho \lambda)_i = G_{i-1} \Delta \lambda_{i-1} - G_i \Delta \lambda_i, \quad \lambda_{i+n} = \rho \lambda_i.$$

Lemma 3.5.

$$\det \mathbf{G}_\rho = (-1)^k \rho^{-k} (\rho - 1)^{2k} \prod_{i=1}^n \det G_i. \quad (3.41)$$

⁶We do not need assumption B in this section.

This follows from Hill's formula (2.18) applied to the DLS (F, Λ_F) with the bilinear action form $h|_Z$. The corresponding Poincaré map P_F has the matrix of the form

$$P_F \sim \begin{pmatrix} I & \overline{G} \\ 0 & I \end{pmatrix}.$$

Hence $\det(P - \rho I) = (\rho - 1)^{2k}$. The operators B_i for the system (F, Λ_F) are equal to the G_i .

Proposition 3.12. *Let $\rho \in S^1$, $\rho \neq 1$. Then $X_\rho = Y_\rho \oplus Z_\rho$ and $Y_\rho \cap Z_\rho = \{0\}$.*

Proof. Let $\mathbf{u} \in X_\rho$. If we want to find $(\lambda_{\alpha i})$, $\lambda_{\alpha i+n} = \rho \lambda_{\alpha i}$, such that

$$(u_i + \lambda_{\alpha i} w_i^\alpha) \in Y_\rho, \quad (3.42)$$

then (3.26) gives

$$\lambda_{\alpha n+1} - \lambda_{\alpha 1} = \sum_{i=1}^n g_{\alpha \beta i} I_i^\beta(u_i, u_{i+1}).$$

Thus

$$\lambda_{\alpha 1} = (\rho - 1)^{-1} \sum_{i=1}^n g_{\alpha \beta i} I_i^\beta(u_i, u_{i+1}).$$

Similarly we find $\lambda_{\alpha 2}, \dots, \lambda_{\alpha n}$. □

Formula (3.42) defines a projection $\Phi_\rho: X_\rho \rightarrow Y_\rho$, $\Phi_\rho Z_\rho = 0$, $\Phi_\rho|_{Y_\rho} = I$. We have

$$\text{ind } h_\rho = \text{ind } h_\rho|_{Z_\rho} + \text{ind } h_\rho|_{Y_\rho}.$$

The projection $\Pi: X_\rho \rightarrow X_\rho^\perp$ gives an isomorphism of $h_\rho|_{Y_\rho}$ and the ρ -index form h_ρ^\perp for the reduced system (E^\perp, Λ^\perp) . Lemma 3.5 implies that the Hill ρ -determinants for the original and the reduced system are related by

$$\det H_\rho = \det H_\rho^\perp \det \mathbf{G}_\rho = \det H_\rho^\perp (2 - \rho - \rho^{-1})^k \prod_{i=1}^n \det G_i.$$

Hence

$$\text{ind } h_\rho|_{Z_\rho} = \sum_{i=1}^n \text{ind } G_i \pmod{2}.$$

Corollary 3.6. *If $\rho \in S^1$, $\rho \neq 1$, then*

$$\text{ind } h_\rho = \text{ind } h_\rho^\perp + \sum_{i=1}^n \text{ind } G_i \pmod{2}, \quad \text{null } h_\rho = \text{null } h_\rho^\perp.$$

4 Reversible version

4.1 Reversible DLS

Let $S: M \rightarrow M$ be a smooth involution: $S^2 = \text{id}_M$. We say that a DLS is S -reversible if S is a time reversing symmetry for L : for any $x, y \in M$,

$$L(Sx, Sy) = L(y, x). \quad (4.1)$$

Equivalently, the Lagrangian L is invariant under the involution $\tilde{S}: M^2 \rightarrow M^2$, $\tilde{S}(x, y) = (Sy, Sx)$.

The simplest example is $S = \text{id}$, that is, $L(x, y) = L(y, x)$ (for example, a billiard system or a standard map). A non-trivial S appears in the system (2.5) if the potential is even. Then $S(x) = -x$. An analogous possibility exists in billiards with some symmetry conditions.

Proposition 4.1. *Suppose that $T: M^2 \rightarrow M^2$ is generated by an S -reversible DLS and $T(x, y) = (y, z)$. Then*

$$T(S(z), S(y)) = (S(y), S(x)).$$

Hence T is conjugate to T^{-1} :

$$T^{-1} \circ \tilde{S} = \tilde{S} \circ T. \quad (4.2)$$

The proof follows by differentiating the identity

$$L(x, y) + L(y, z) = L(Sz, Sy) + L(Sy, Sx).$$

If $\mathbf{x} = (x_i)$ is a periodic orbit of a DLS, then $\tilde{\mathbf{x}} = (Sx_{-i})$ is also a periodic orbit. A periodic orbit is called reversible if $\tilde{\mathbf{x}} = \mathbf{x}$ modulo translations. The group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acts on the set of n -periodic sequences $(x_i)_{i \in \mathbb{Z}}$ in M by translation $(x_i)_{i \in \mathbb{Z}} \mapsto (x_{i+k})_{i \in \mathbb{Z}}$, and we should identify periodic orbits obtained in such a way. Any n -periodic sequence is determined by $(x_1, \dots, x_n) \in M^n$ and the translation group \mathbb{Z}_n acts on M^n by cyclic permutations. Thus the set of periodic sequences is the quotient $\mathcal{M} = M^n / \mathbb{Z}_n$. Define an involution $R: \mathcal{M} \rightarrow \mathcal{M}$ by $R(\mathbf{x}) = (Sx_{-i})$. Let $\mathcal{M}_+ = \{\mathbf{x} \in \mathcal{M} : R\mathbf{x} = \mathbf{x}\}$ be the set of fixed points of R . Thus $\mathbf{x} \in \mathcal{M}_+$ if and only if $S(x_{j-i}) = x_i$ for some $j \in \mathbb{Z}$ and all $i \pmod{n}$.

Proposition 4.2. *$\mathbf{x} \in \mathcal{M}_+$ is a reversible periodic orbit if and only if \mathbf{x} is a critical point of the functional $\mathcal{A}_+ = \mathcal{A}|_{\mathcal{M}_+}$.*

This is a well known property of functions invariant under an involution. Indeed, let $X = T_{\mathbf{x}}\mathcal{M}$. Then $J = dR(\mathbf{x}): X \rightarrow X$ is an involution. Denote $E_{\pm} = \{\xi \in X :$

$J\xi = \pm\xi\}$. Then $X = E_+ \oplus E_-$ and $E_+ = T_{\mathbf{x}}\mathcal{M}_+$. Since \mathcal{A} is R -invariant, we have $d\mathcal{A}(\mathbf{x}) \circ J = d\mathcal{A}(\mathbf{x})$. Thus, $d\mathcal{A}(\mathbf{x})\xi = 0$ for all $\xi \in E_-$.

For any critical point $\mathbf{x} \in \mathcal{M}_+$ let $h = d^2\mathcal{A}(\mathbf{x})$ be the Hessian, that is, the second differential of \mathcal{A} . This is a bilinear form on X . Then $h|_{E_+} = d^2\mathcal{A}_+(\mathbf{x})$ is the second differential of \mathcal{A}_+ . For $\xi = \xi_+ + \xi_-$, $\xi_{\pm} \in E_{\pm}$, we obtain

$$h(\xi, \xi) = h(\xi_+, \xi_+) + h(\xi_-, \xi_-).$$

Indeed, since h is J -invariant,

$$h(\xi_-, \xi_+) = h(J\xi_-, J\xi_+) = h(-\xi_-, \xi_+) = -h(\xi_-, \xi_+).$$

If we represent h by a linear operator $\mathbf{H}: X \rightarrow X^*$, then $J^*\mathbf{H}J = \mathbf{H}$, and so $\mathbf{H}E_{\pm} = E_{\pm}^*$.

Let us introduce on M an S -invariant Riemannian metric. It defines an R -invariant metric (\cdot, \cdot) on X . Then $h(\xi, \eta) = (H\xi, \eta)$, where $H: X \rightarrow X$ is a symmetric operator. The spaces E_{\pm} are orthogonal with respect to the metric and $HE_{\pm} \subset E_{\pm}$. Denote $H_{\pm} = H|_{E_{\pm}}$. We obtain

Proposition 4.3. $H = H_+ \oplus H_-$ and $\det H = \det H_+ \det H_-$.

Reversible periodic trajectories $\mathbf{x} \in \mathcal{M}_+$ are of 3 types $\tau = 0, 1, 2$ depending on the number of fixed points of S they contain.

Type 0: $n = 2k$ and $\mathbf{x} = (x_1, \dots, x_k, Sx_k, \dots, Sx_1)$.

Type 1: $n = 2k - 1$ and $\mathbf{x} = (x_1, \dots, x_k, Sx_k, \dots, Sx_2)$, where $x_1 = Sx_1$.

Type 2: $n = 2k - 2$ and $\mathbf{x} = (x_1, \dots, x_k, Sx_{k-1}, \dots, Sx_2)$, where $x_1 = Sx_1$ and $x_k = Sx_k$.

For all types $\mathbf{x} = \mathbf{x}_{\tau}(\mathbf{y}) \in \mathcal{M}_+$ is determined by $\mathbf{y} = (x_1, \dots, x_k) \in M^k$. Thus the action functional \mathcal{A}_+ on \mathcal{M}_+ gives a function on M^k :

$$\mathcal{A}_{\tau}(\mathbf{y}) = \mathcal{A}_+(\mathbf{x}_{\tau}(\mathbf{y})) = \mathcal{A}_+(\mathbf{x}), \quad \tau = 0, 1, 2.$$

Denote

$$\mathcal{B}(\mathbf{y}) = \sum_{i=1}^{k-1} L(x_i, x_{i+1}).$$

Lemma 4.1. $\mathbf{x} = \mathbf{x}_{\tau}(\mathbf{y})$ is a periodic orbit of type $\tau = 0, 1, 2$ if and only if \mathbf{y} is a critical point of

$$\begin{aligned} \mathcal{A}_0(\mathbf{y}) &= 2\mathcal{B}(\mathbf{y}) + L(Sx_1, x_1) + L(x_k, Sx_k), & \mathbf{y} &\in M^k, \\ \mathcal{A}_1(\mathbf{y}) &= 2\mathcal{B}(\mathbf{y}) + L(x_k, Sx_k), & \mathbf{y} &\in N \times M^{k-1}, \\ \mathcal{A}_2(\mathbf{y}) &= 2\mathcal{B}(\mathbf{y}), & \mathbf{y} &\in N \times M^{k-2} \times N, \end{aligned}$$

respectively.

The functional \mathcal{A} on \mathcal{M} admits a similar representation. For example, consider the case of periodic orbits of type 0. A point $\mathbf{x} \in M^n$ can be written as

$$\mathbf{x} = (y_1, \dots, y_k, Sz_k, \dots, Sz_1).$$

Then, since $L(Sz_{i+1}, Sz_i) = L(z_i, z_{i+1})$,

$$\mathcal{A}(\mathbf{x}) = \mathcal{B}(\mathbf{y}) + \mathcal{B}(\mathbf{z}) + L(y_k, Sz_k) + L(Sz_1, y_1).$$

For \mathbf{x} a periodic orbit of type 0 we have $\mathbf{y} = \mathbf{z} = (x_1, \dots, x_k)$. We write $\mathbf{u} \in T_{\mathbf{x}}M^n$ as

$$\mathbf{u} = (v_1, \dots, v_k, J_k w_k, \dots, J_1 w_1), \quad \mathbf{v}, \mathbf{w} \in T_{\mathbf{y}}M^k,$$

where $J_i = dS(x_i): T_{x_i}M \rightarrow T_{Sx_i}M$. Taking the second differential of \mathcal{A} we get

$$h(\mathbf{u}, \mathbf{u}) = k(\mathbf{v}, \mathbf{v}) + k(\mathbf{w}, \mathbf{w}) - \langle B_k^* J_k w_k, v_k \rangle - \langle v_1, B_0 J_1 w_1 \rangle,$$

where

$$k(\mathbf{v}, \mathbf{v}) = \sum_{i=1}^k \langle A_i v_i - B_{i-1} v_{i-1} - B_i^* v_{i+1}, v_i \rangle, \quad v_0 = 0, \quad v_{k+1} = 0,$$

and $B_k = B(x_k, Sx_k)$, $B_0 = B(Sx_1, x_1)$. Denote

$$C_1 = -B_0 J_1: E_1 \rightarrow E_1^*, \quad C_k = -B_k^* J_k: E_k \rightarrow E_k^*.$$

Note that $C_1 = C_1^*$ and $C_k = C_k^*$ are symmetric.. Thus

$$h(\mathbf{u}, \mathbf{u}) = k(\mathbf{v}, \mathbf{v}) + k(\mathbf{w}, \mathbf{w}) + \langle C_k v_k, w_k \rangle + \langle C_1 v_1, w_1 \rangle. \quad (4.3)$$

Let us compute the corresponding bilinear forms $h|_{E_{\pm}}$. For $\mathbf{u} \in E_{\pm}$ we have $\mathbf{w} = \pm \mathbf{v}$, so \mathbf{u} is determined by \mathbf{v} :

$$h_{\pm}(\mathbf{u}, \mathbf{u}) = h_{\pm}^0(\mathbf{v}, \mathbf{v}) = 2k(\mathbf{v}, \mathbf{v}) \pm \langle C_k v_k, v_k \rangle \pm \langle C_1 v_1, v_1 \rangle.$$

Similarly, for any $\tau = 0, 1, 2$ a vector $\mathbf{u} \in E_{\pm}$ is determined by $\mathbf{v} = (v_1, \dots, v_k) \in T_{\mathbf{y}}M$.

Lemma 4.2. *For a reversible orbit of type $\tau = 0, 1, 2$, $h_{\pm}(\mathbf{u}, \mathbf{u}) = h_{\pm}^{\tau}(\mathbf{v}, \mathbf{v})$ has the form*

$$\begin{aligned} h_{\pm}^0(\mathbf{v}, \mathbf{v}) &= 2k(\mathbf{v}, \mathbf{v}) \pm \langle C_k v_k, v_k \rangle \pm \langle C_1 v_1, v_1 \rangle, \\ h_{\pm}^1(\mathbf{v}, \mathbf{v}) &= 2k(\mathbf{v}, \mathbf{v}) \pm \langle C_k v_k, v_k \rangle, \\ h_{\pm}^2(\mathbf{v}, \mathbf{v}) &= 2k(\mathbf{v}, \mathbf{v}). \end{aligned}$$

The domain of h_{\pm}^{τ} is V_{\pm}^{τ} , where

$$\begin{aligned} V_{\pm}^0 &= T_{\mathbf{y}}M^k, \\ V_{\pm}^1 &= \{\mathbf{v} \in T_{\mathbf{y}}M^k : J_1 v_1 = \pm v_1\}, \\ V_{\pm}^2 &= \{\mathbf{v} \in T_{\mathbf{y}}M^k : J_1 v_1 = \pm v_1, J_k v_k = \pm v_k\}. \end{aligned}$$

Consider a periodic orbit of type 0. Then the domains of h_+^0 and h_-^0 coincide and

$$h_+^0(\mathbf{v}, \mathbf{v}) - h_-^0(\mathbf{v}, \mathbf{v}) = 2\langle C_1 v_1, v_1 \rangle + 2\langle C_k v_k, v_k \rangle.$$

Corollary 4.1. *Suppose that $\mathbf{x} \in \mathcal{M}_+$ is a periodic orbit of type 0 which is a nondegenerate local minimum point of \mathcal{A}_+ . If the symmetric operators C_1 and C_k are non-positive, then \mathbf{x} is a nondegenerate local minimum for \mathcal{A} . If $\mathbf{x} \in \mathcal{M}_+$ is a nondegenerate local maximum point of \mathcal{A}_+ and C_1, C_k are non-negative, then \mathbf{x} is a nondegenerate local maximum for \mathcal{A} . In both cases $\det H = \det H_- \det H_+ > 0$.*

For periodic orbits of type $\tau = 1$ or 2 , the domains of h_+^{τ} and h_-^{τ} are different. When $S = \text{id}$, then $V_-^{\tau} \subset V_+^{\tau}$.

Proposition 4.4. *Suppose that $S = \text{id}_M$, and let \mathbf{x} be a reversible periodic trajectory of type τ which is a nondegenerate minimum for \mathcal{A}_{τ} . Then in each of the three cases*

- (a) $\tau = 2$,
- (b) $\tau = 1$ and C_k is non-positive,
- (c) $\tau = 0$ and C_k, C_1 are non-positive

\mathbf{x} is a nondegenerate minimum for \mathcal{A} .

If \mathbf{y} is a nondegenerate maximum for \mathcal{A}_{τ} , then a similar statement holds provided that C_k and C_1 are non-negative rather than non-positive.

Proposition 4.5. *Let $S = \text{id}$. If \mathcal{A}_+ is a minimal periodic orbit of type 2, then h_{ρ} is positive definite for all $\rho \in S^1$. Hence \mathbf{x} is hyperbolic.*

Proof. Take complex $\mathbf{u} \in T_{\mathbf{x}}^{\mathbb{C}}M^{2k}$. Then

$$h_{\rho}(\mathbf{u}, \bar{\mathbf{u}}) = k(\mathbf{u}^+, \bar{\mathbf{u}}^+) + k(\mathbf{u}^-, \bar{\mathbf{u}}^-),$$

where $\mathbf{u}^+ = (u_1, \dots, u_k)$ and $\mathbf{u}^- = (J_{2k}u_{2k}, \dots, J_{k+1}u_{k+1})$ are complex vectors from $T_{\mathbf{y}}^{\mathbb{C}}M^k$. Since k is positive definite, h_{ρ} is positive definite. \square

4.2 Some applications

Corollary 4.2. *Reversible geometric degeneracy $\det H_+ = 0$ of a reversible trajectory implies the dynamical degeneracy $\det(P - I) = 0$.*

Next we give some statements on dynamical stability (in fact, instability) of reversible trajectories.

Corollary 4.3. *Let \mathbf{x} be a reversible periodic trajectory such that \mathbf{y} is a nondegenerate minimum point of \mathcal{A}_+ . Suppose also that $\sigma(\mathbf{x})(-1)^m < 0$. In the case $\text{type}(\mathbf{x}) = 1$ we also need the condition that C_k is non-positive, and in the case $\text{type}(\mathbf{x}) = 0$ that C_1 and C_k are non-positive. Then \mathbf{x} has a real multiplier > 1 . In particular \mathbf{x} is dynamically unstable.*

Proof. By Corollary 4.1, both h_\pm are positive definite. Therefore $\det H_\pm > 0$. Now by (2.8) and Proposition 4.3 we have $\det(P - I) = -1$. It remains to use Proposition 2.2. \square

Consider, for example, the DLS generated by billiards in a domain in \mathbb{R}^{m+1} bounded by a hypersurface M . Suppose that the billiard hypersurface M is symmetric with respect to a hyperplane in \mathbb{R}^{m+1} , for definiteness passing through 0. Then the symmetry $S = S_e: \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$ is given by $S_e(x) = x - 2e\langle x, e \rangle$, where e is the unit normal vector.

Proposition 4.6. *Let $x, y \in M$, $y = S_e x$, be a pair of symmetric points and let $B = B(y, x)$. Then the operator $C = -BS_e: T_y M \rightarrow T_y^* M$ is symmetric and positive definite: $\langle Cv, v \rangle < 0$ for non-zero $v \in T_y M$.*

Indeed, by (2.6),

$$\langle Cv, w \rangle = \frac{-\langle S_e v, w \rangle + \langle S_e v, e \rangle \langle w, e \rangle}{|x - y|} = -\frac{\langle v - e\langle v, e \rangle, w - e\langle w, e \rangle \rangle}{|x - y|},$$

where we used $S_e v = v - 2e\langle v, e \rangle$.

Corollary 4.4. *Let \mathbf{x} be an S_e -reversible periodic billiard trajectory of type τ such that \mathbf{y} is a nondegenerate minimum of the length functional \mathcal{A}_+ . If $m + \tau$ is odd, then \mathbf{x} has a real multiplier greater than 1.*

Proof. Consider the case $\tau = 0$. By Proposition 4.6, the symmetric operators C_1 and C_k are negative definite. Therefore by Proposition 4.4, \mathbf{x} is a nondegenerate minimum of \mathcal{A} . Note also that $\sigma(\mathbf{x}) = (-1)^n = (-1)^\tau > 0$. Now it remains to use Corollary 2.6. The cases $\tau = 1, 2$ are analogous. \square

Any billiard is S -reversible for $S = \text{id}$. Any reversible periodic trajectory \mathbf{x} is of type 2. By Proposition 2.1, $\sigma(\mathbf{x}) = (-1)^n > 0$.

Corollary 4.5. *Any id-reversible billiard trajectory which gives a nondegenerate minimum of the functional \mathcal{A}_+ , is hyperbolic.*

This follows from Proposition 4.5.

5 Hill's formula for a continuous Lagrangian system

Consider a continuous Lagrangian system (M, \mathcal{L}) with the configuration space M^m and smooth⁷ τ -periodic Lagrangian $\mathcal{L}(x, \dot{x}, t)$ on $TM \times \mathbb{R}$. We assume that \mathcal{L} is strictly convex in velocity $\dot{x} \in T_x M$. Then τ -periodic trajectories are critical points of the action functional

$$\mathcal{A}(\gamma) = \int_0^\tau \mathcal{L}(\gamma(t), \dot{\gamma}(t), t) dt$$

on the space Ω of τ -periodic $W^{1,2}$ curves $\gamma: \mathbb{R} \rightarrow M$. The goal of this section is to prove an analogue of Theorem 2.1 for continuous Lagrangian systems.

5.1 Continuous Hill determinant

The second variation of the functional \mathcal{A} at $\gamma \in \Omega$ is a symmetric bilinear form $h(\xi, \eta)$ on the set X of τ -periodic $W^{1,2}$ vector fields $\xi(t) \in E_t = T_{\gamma(t)}M$ along γ . It is defined by

$$h(\xi, \xi) = \left. \frac{d^2}{d\alpha^2} \right|_{\alpha=0} \mathcal{A}(\gamma_\alpha), \quad \gamma_0 = \gamma,$$

where $\gamma_\alpha: \mathbb{R} \rightarrow M$ is a smooth τ -periodic variation of γ . Define a positive definite scalar product on E_t by

$$(v, w) = \langle B(t)v, w \rangle, \quad B(t) = \mathcal{L}_{\dot{x}\dot{x}}(\gamma(t), \dot{\gamma}(t), t).$$

Proposition 5.1. *h can be uniquely represented in the form*

$$h(\xi, \eta) = \int_0^\tau \left((D\xi(t), D\eta(t)) + (U(t)\xi(t), \eta(t)) \right) dt, \quad (5.1)$$

where $U(t) = U^*(t)$ is a symmetric linear operator and D is a covariant derivative, that is, a linear differential operator such that

$$\frac{d}{dt}(\xi, \eta) = (D\xi, \eta) + (\xi, D\eta), \quad \frac{d}{dt}(f\xi) = \dot{f}\xi + f D\xi \quad (5.2)$$

for smooth vector fields $\xi(t), \eta(t) \in E_t$ and a scalar function $f(t)$.

⁷Actually, C^2 is enough.

Proof. Let ∇ be any covariant derivative.⁸ A standard computation shows that h can be written in the form

$$h(\xi, \eta) = \int_0^\tau \left(((\nabla \xi(t), \nabla \eta(t)) + (W(t)\xi(t), \nabla \eta(t)) + (V(t)\xi(t), \eta(t))) \right) dt,$$

where $V(t), W(t): E_t \rightarrow E_t$ are linear operators, and $V(t)$ is symmetric with respect to the metric: $V(t) = V^*(t)$.

By integration by parts h can be represented in the form (5.1), where

$$D\xi = \nabla \xi + W - W^*, \quad U = V - W - W^* = U^*.$$

Hence D is also a covariant derivative. \square

Note that D and U are invariantly determined by h , that is, they are coordinate independent and do not change by a calibration of the Lagrangian.

Equations (5.2) imply that D is skew-symmetric relative to the L^2 scalar product

$$(\xi, \eta)_2 = \int_0^\tau (\xi(t), \eta(t)) dt.$$

Therefore

$$h(\xi, \eta) = ((-D^2 + U)\xi, \eta)_2 = (\mathbf{H}\xi, \eta)_2, \quad (5.3)$$

where $\mathbf{H} = -D^2 + U$ is the Hessian of \mathcal{A} with respect to the L^2 -metric.

The variational system of the periodic trajectory γ has the form

$$D^2\xi(t) = U(t)\xi(t). \quad (5.4)$$

This is a linear Lagrangian system. We use the following definition.

Definition 5.1. Let $E = \{E_t\}_{t \in \mathbb{R}/\tau\mathbb{Z}}$ be a smooth vector bundle. Suppose it is equipped with a metric (\cdot, \cdot) compatible with a covariant derivative D and a symmetric linear operator $U(t): E_t \rightarrow E_t$. Denote by (E, Λ) the linear Lagrangian system with the quadratic Lagrangian

$$\Lambda(\xi, D\xi) = \frac{1}{2} (D\xi, D\xi) + \frac{1}{2} (U\xi, \xi) \quad (5.5)$$

and Lagrange's equations (5.4).

⁸ A covariant derivative is not uniquely defined: for an antisymmetric operator $A(t)$, $\nabla + A(t)$ is also a covariant derivative. We use a covariant derivative because the derivative is undefined unless E_t is t -independent.

Trajectories $\xi(t)$, $0 \leq t \leq \tau$, of the system (E, Λ) are extremals of the quadratic action functional

$$\frac{1}{2} h(\xi, \xi) = \int_0^\tau \Lambda(\xi, D\xi) dt \quad (5.6)$$

for variations with fixed $\xi(0), \xi(\tau)$. Thus $h(\xi, \phi) = 0$ for any smooth $\phi(t) \in E_t$ such that $\phi(0) = 0$ and $\phi(\tau) = 0$.

The system (E, Λ) is the linearization of (M, \mathcal{L}) at γ . In what follows we can forget about the non-linear Lagrangian system (M, \mathcal{L}) and work with the linear system (E, Λ) .

Let $P : W \rightarrow W$ be the linear Poincaré map of the trajectory γ . Since a solution $\xi(t)$ of the variational system is uniquely determined by $(\xi(0), D\xi(0))$, W can be identified with $E_0 \oplus E_0$. Then P is the monodromy operator of the variational system:

$$P(\xi(0), D\xi(0)) = (\xi(\tau), D\xi(\tau)).$$

Define the $W^{1,2}$ -scalar product on the Hilbert space X by

$$\langle\langle \xi, \eta \rangle\rangle = (D\xi, D\eta)_2 + (\xi, \eta)_2 = (\mathbf{B}\xi, \eta)_2, \quad \mathbf{B} = -D^2 + I.$$

Then $h(\xi, \bar{\eta}) = \langle\langle H\xi, \bar{\eta} \rangle\rangle$, where the self-adjoint operator $H = \mathbf{B}^{-1}\mathbf{H}$ is the Hessian of \mathcal{A} with respect to the $W^{1,2}$ -scalar product.

We have $H = I + K$, where $K = (-D^2 + I)^{-1}(U - I)$ is compact, with eigenvalues $\lambda_k = O(k^{-2})$, $k = 1, 2, \dots$, so that

$$\text{tr } |K| = \sum_{k=1}^{\infty} |\lambda_k| < \infty.$$

Thus, the *Hill determinant*

$$\det H = \prod_{k=1}^{\infty} (1 + \lambda_k)$$

converges absolutely.

Let $Q : E_0 \rightarrow E_0$ be the monodromy operator of the equation of parallel transport:

$$Q\eta(0) = \eta(\tau), \quad D\eta(t) = 0.$$

Theorem 5.1. $\det(I - P) = \sigma(-1)^m \beta \det H$, where

$$\beta = e^{-m\tau} \det^2(e^\tau I - Q), \quad \sigma = \det Q. \quad (5.7)$$

Since Q is an orthogonal operator, $\beta > 0$ and $\sigma = \pm 1$ depending on whether the bundle E is orientable, that is, if the trajectory γ preserves or reverses orientation. If M is orientable, then $\sigma = 1$ always.

Theorem 5.1 follows from a more general result of the next subsection.

Example 5.1. Suppose the Lagrangian system has one degree of freedom and the bundle E is trivial. Then $Q = 1$ and $\mathbf{H}\xi = -\ddot{\xi} + a(t)\xi$. Since $\det P = 1$,

$$\rho^{-1} \det(\rho I - P) = \rho + \rho^{-1} - 2 + \det(I - P).$$

If ρ is a multiplier, Theorem 5.1 gives

$$\det H = \frac{\rho + \rho^{-1} - 2}{e^\tau + e^{-\tau} - 2}. \quad (5.8)$$

Set $\tau = 2\pi$ and represent the operator H in the basis $\{e^{int}\}$. If

$$a(t) = \sum_{n \in \mathbb{Z}} a_n e^{int}, \quad \xi(t) = \sum_{n \in \mathbb{Z}} \xi_n e^{int},$$

then

$$H\xi = (-D^2 + I)^{-1}(-D^2 + U)\xi = \sum_{n \in \mathbb{Z}} (n^2 + 1)^{-1} \left(n^2 + \sum_{k \in \mathbb{Z}} a_k e^{ikt} \right) \xi_n e^{int}.$$

Hence (1.3) is the matrix of H , and (5.8) gives Hill's formula (1.2).

5.2 Relation to Hill's formula for discrete Lagrangian systems

A continuous Lagrangian system locally, near a periodic orbit γ , defines a discrete Lagrangian system. Take a partition $0 = t_0 < t_1 < \cdots < t_n = \tau$ of $[0, \tau]$ and let $x_i = \gamma(t_i)$. If the $\Delta t_i = t_{i+1} - t_i$ are small enough, the points x_i, x_{i+1} are non-conjugate along γ . Then there is a neighbourhood U_i of (x_i, x_{i+1}) in M^2 such that for each $(x, y) \in U_i$ there exists a unique trajectory $u_{x,y}: [t_i, t_{i+1}] \rightarrow M$ close to $\gamma|_{[t_i, t_{i+1}]}$ and joining x and y . Define a discrete Lagrangian L_i on U_i by

$$L_i(x, y) = \mathcal{A}(u_{x,y}).$$

The discrete action functional A is

$$A(\mathbf{y}) = \sum_{i=1}^n L_i(y_i, y_{i+1}), \quad (y_i, y_{i+1}) \in U_i, \quad y_{n+1} = y_1.$$

Then \mathbf{x} is a critical point of A , that is, the periodic orbit of the DLS corresponding to the periodic orbit γ of the continuous Lagrangian system. It is easy to see that

$$B_i = -\partial_{12}L_i(x_i, x_{i+1}) = \frac{1}{\Delta t_i}(B(t_i) + O(\Delta t_i)).$$

Thus L_i is a discrete Lagrangian.

The definition of the Hill determinant $\det H = \det(\mathbf{B}^{-1}\mathbf{H})$ for a CLS is similar to (2.8) for a DLS. Discretization of the operator $\mathbf{H} = -D^2 + U$ corresponds to the operator \mathbf{H} in (2.8). However, the operator $\mathbf{B} = -D^2 + I$ does not correspond to the operator \mathbf{B} in (2.11).

The choice of \mathbf{B} is natural for DLS, but not so for CLS, where instead of I we could add almost anything. This is the reason for the strange coefficient β in (5.7). If we use an analogue of discrete \mathbf{B} , then $\mathbf{B}^{-1}\mathbf{H}$ will be unbounded.

5.3 Generalized Hill determinant

For a given $\rho \in S^1$ let X_ρ be the vector space of complex ρ -quasiperiodic locally $W^{1,2}$ vector fields $\xi(t) \in E_t$ such that $\xi(t + \tau) = \rho\xi(t)$. Define a Hermitian ρ -index form [28], [27] on X_ρ by (5.1):

$$h(\xi, \bar{\eta}) = \int_0^\tau \left((D\xi(t), D\bar{\eta}(t)) + (U(t)\xi(t), \bar{\eta}(t)) \right) dt. \quad (5.9)$$

We also denote by X the complexification $X = X^\mathbb{C}$, that is, the set of complex τ -periodic $W^{1,2}$ -vector fields along γ . For definiteness choose $\ln \rho$ so that

$$0 \leq \operatorname{Im} \ln \rho < 2\pi,$$

and let $\mu = \tau^{-1} \ln \rho$. Identifying X and X_ρ by the map

$$X \ni \xi \mapsto e^{\mu t} \xi(t) \in X_\rho, \quad (5.10)$$

we obtain a Hermitian form h_ρ on X :

$$\begin{aligned} h_\rho(\xi, \bar{\eta}) &= h(e^{\mu t} \xi, \overline{e^{\mu t} \eta}) = ((D + \mu I)\xi, \overline{(D + \mu I)\eta})_2 + (U\xi, \bar{\eta})_2 \\ &= -((D + \mu I)^2 \xi, \bar{\eta})_2 + (U\xi, \bar{\eta})_2 = (-(D + \mu I)^2 + U)\xi, \bar{\eta})_2 \\ &= (\mathbf{H}_\rho \xi, \bar{\eta})_2, \quad \text{where } \mathbf{H}_\rho = -(D + \mu I)^2 + U. \end{aligned}$$

(We used that $\bar{\mu} = -\mu$ and D is real and antisymmetric). Define the ρ -Hessian operator $H_\rho: X \rightarrow X$ by $h_\rho(\xi, \bar{\eta}) = \langle \langle H_\rho \xi, \bar{\eta} \rangle \rangle$. Then

$$H_\rho = \mathbf{B}^{-1}\mathbf{H}_\rho = (-D^2 + I)^{-1}(-(D + \mu I)^2 + U). \quad (5.11)$$

We henceforth assume that $\rho \neq 0$ may take any complex values. The operator H_ρ is self adjoint for $|\rho| = 1$, but not in general. Although this is not a trace class operator: $\text{tr} |H_\rho - I|$ diverges for $\rho \neq 1$, we can, following Poincaré [2], define the generalized Hill determinant $\det H_\rho$ by means of the finite-dimensional approximation

$$\det H_\rho = \lim_{N \rightarrow \infty} \det H_\rho^{(N)}, \quad H_\rho^{(N)} = P_N H_\rho P_N^*: X^{(N)} \rightarrow X^{(N)}, \quad (5.12)$$

where P_N is the orthogonal projection onto the finite-dimensional eigenspace $X^{(N)}$ of the operator D corresponding to the eigenvalues in

$$\Lambda_N = \{\nu \in \Lambda = \sigma(D) : |\nu| \leq N\}.$$

Theorem 5.2. *The determinant (5.12) converges and*

$$\rho^{-m} \det(P - \rho I) = \sigma(-1)^m \beta \det H_\rho. \quad (5.13)$$

We present the proof of Theorem 5.2 in § 5.5.

For $\rho = 1$ we obtain Theorem 5.1.

5.4 Some applications

Suppose that $\mathcal{L}(x, \dot{x}) = (\dot{x}, \dot{x})/2$, where (\cdot, \cdot) is a Riemannian metric on M . The periodic orbit γ is a closed geodesic. The quadratic Lagrangian of the variational system has the form (5.5), where D is the Levi-Civita covariant derivative along γ and $U(t)\xi = R(\xi, \dot{\gamma}(t))\xi$ with R the curvature tensor.

The variational system has a periodic solution $\dot{\gamma}(t)$ and a first integral $(D\xi, \dot{\gamma}) = \frac{d}{dt}(\xi, \dot{\gamma})$ periodic in time. Hence P has two unit multipliers, and $\det H = \det(I - P) = 0$. Let us present a reduced version of Hill's formula. More general results will be proved in the next subsection (see Corollary 6.1).

Let $E_t^\perp = \{u \in E_t : (u, \dot{\gamma}(t)) = 0\}$. If $\xi(t) \in E_t^\perp$, then $D\xi(t), U(t)\xi(t) \in E_t^\perp$. Denote by $H^\perp: X^\perp \rightarrow X^\perp$ the restriction of H to the invariant subspace

$$X^\perp = \{\xi \in X : \xi(t) \in E_t^\perp\}.$$

Let $P^\perp: W^\perp \rightarrow W^\perp$, $W^\perp = E_0^\perp \times E_0^\perp$, be the monodromy operator corresponding to solutions $\xi(t) \in E_t^\perp$ of the variational system. Let $Q^\perp: E_0^\perp \rightarrow E_0^\perp$ be the map of parallel transport along γ . Applying Theorem 5.1 to the linear Lagrangian system (E^\perp, Λ^\perp) , where $\Lambda^\perp = \Lambda|_{E^\perp}$, we obtain the following result [5].

Corollary 5.1. *Hill's formula for the reduced system has the form*

$$\det(P^\perp - I) = \sigma(-1)^{m-1} \beta^\perp \det H^\perp, \quad \beta^\perp = e^{-(m-1)\tau} \det^2(Q^\perp - e^\tau I).$$

Let us formulate another corollary to Theorem 5.2. For complex ρ let X^\perp be the space of complex vector fields $\xi(t) \in E_t$ and let $H_\rho^\perp = H_\rho|_{X^\perp}$. Then

$$\rho^{-(m-1)} \det(P^\perp - \rho I) = \sigma(-1)^{m-1} \beta^\perp \det H_\rho^\perp. \quad (5.14)$$

Let us present a proof of (5.14) from [5], which will be generalized in § 6.1. Let $Z = \{\xi \in X : \xi(t) = \lambda(t)\dot{\gamma}(t)\}$. We write any $\xi \in X$ as $\xi(t) = \eta(t) + \lambda(t)\dot{\gamma}(t)$, where $\eta \in X^\perp$. Then

$$h(\xi, \bar{\xi}) = h(\eta, \bar{\eta}) + \int_0^\tau |\dot{\lambda}|^2 dt. \quad (5.15)$$

Hence $H_\rho = H_\rho|_Z \oplus H_\rho^\perp$ and $\det H_\rho = \det H_\rho|_Z \det H_\rho^\perp$, where $\det H_\rho|_Z$ is the Hill determinant for the system with the quadratic Lagrangian $|\dot{\lambda}|^2/2$. The characteristic polynomial of the corresponding monodromy matrix is $(\rho - 1)^2$. Thus by (5.8),

$$\det H_\rho|_Z = -\frac{e^\tau(\rho - 1)^2}{\rho(e^\tau - 1)^2}.$$

But

$$\det(\rho I - P) = (\rho - 1)^2 \det(\rho I - P^\perp), \quad \det(e^\tau I - Q) = (e^\tau - 1) \det(e^\tau I - Q^\perp),$$

which implies (5.14).

Next we discuss applications to stability of periodic trajectories, similar to the discrete case. For $\rho \in S^1$ define the ρ -index ([28], [27]) $\text{ind}_\rho \gamma$ of a periodic trajectory γ as the index of the Hermitian form h_ρ . Then $\text{ind} \gamma = \text{ind}_1 \gamma$ is the Morse index of γ . It equals the number of negative eigenvalues of the operator H . If ρ is not an eigenvalue of P ,

$$(-1)^{\text{ind}_\rho \gamma} = \text{sign} \det H_\rho = \sigma(-1)^m \text{sign}(\rho^{-m} \det(\rho I - P)).$$

The argument of the sign function is real for $|\rho| = 1$ since the characteristic polynomial is reciprocal.

The next result is proved in [5].

Corollary 5.2. *Suppose the trajectory γ is nondegenerate and $\sigma(-1)^{m+\text{ind} \gamma} < 0$. Then γ has a real multiplier $\rho > 1$.*

Indeed, the characteristic polynomial $F(\rho) = \det(\rho I - P)$ satisfies $F(1) < 0$ and $F(+\infty) = +\infty$. Hence F has a real root $\rho > 1$.

Corollary 5.2 is not true if γ is degenerate. Suppose for example, that γ is a closed geodesic.⁹ Then (5.15) implies $\text{ind} h_\rho = \text{ind} h_\rho^\perp$, and so

$$(-1)^{\text{ind}_\rho \gamma} = \text{sign} \det H_\rho^\perp = \sigma(-1)^{m-1} \text{sign}(\rho^{1-m} \det(\rho I - P^\perp)).$$

⁹The general degenerate case is discussed in § 6.

Corollary 5.3. *Suppose the closed geodesic γ is nondegenerate and $\sigma(-1)^{m+\text{ind } \gamma} > 0$. Then the characteristic polynomial $F(\rho) = \det(\rho I - P)$ has a real root $\rho > 1$. Therefore, γ is exponentially unstable.*

This is proved in [5] and [29] using Hill's formula and also recently in [11] using the theory of Maslov index [7]. In particular, nondegenerate closed geodesics of locally minimal length on an even-dimensional orientable manifold are exponentially unstable. Degenerate geodesics are linearly unstable, but in general instability will not be exponential and so has no relevance for applications to Lyapunov stability.

Suppose $m = 1$ and let the 2τ -periodic trajectory γ^2 be γ traversed twice. If γ^2 is nondegenerate (that is, ± 1 are not multipliers), then γ has hyperbolic (elliptic) type if and only if $\text{ind } \gamma^2$ is even (odd).

Indeed, γ and γ^2 are simultaneously elliptic or hyperbolic. The multipliers of γ^2 are squares of the multipliers of γ . Hence, γ^2 is hyperbolic if and only if its multipliers are real and positive, or, equivalently, $\text{sign } \det(I - P^2) = (-1)^{1+\text{ind } \gamma^2} = -1$. Similarly, the ellipticity of γ^2 is equivalent to $(-1)^{1+\text{ind } \gamma^2} = 1$.

For the geodesic case, we obtain the following result of Poincaré. Let γ be a closed geodesic on a 2-dimensional Riemannian manifold. If γ^2 is nondegenerate, then γ has hyperbolic (elliptic) type if and only if $\text{ind } \gamma^2$ is even (odd).

Suppose now $\rho = -1$. Then $(-1)^{\text{ind } \gamma} = \sigma \text{sign } F(-1)$. Thus, if $\text{ind } \gamma$ is odd, there exists a real multiplier $\rho < -1$. Note that the space X_{-1} corresponds to antiperiodic variations such that $\xi(\tau) = -\xi(0)$. Since 2τ -periodic vector fields are sums of τ -periodic and τ -antiperiodic,

$$\text{ind}_{-1} \gamma = \text{ind } \gamma^2 - \text{ind } \gamma.$$

Thus, if $\text{ind } \gamma$ and $\text{ind } \gamma^2$ are not even or odd simultaneously, then γ is unstable.

5.5 Proof of Theorem 5.2

We follow [5], see also [6]. The method goes back to Poincaré's proof ([2], [30]) of Hill's result [1].

The real skew-Hermitian operator $D = \overline{D} = -D^*$ has compact resolvent $(D + \mu I)^{-1}$. Its spectrum $\Lambda = \sigma(D) \subset i\mathbb{R}$ coincides with the set of characteristic exponents of the equation $D\eta(t) = 0$ of parallel transport. Thus

$$\Lambda = \{\nu : \det(Q - e^{\tau\nu} I) = 0\}.$$

If $\nu \in \Lambda$, then $-\nu$ and $\nu + \omega$ belong to Λ , where $\omega = 2\pi i/\tau$.

Let ρ_1, \dots, ρ_m be the roots of $\det(Q - \rho I) = 0$. Since $|\rho_j| = 1$, we may represent them as $\rho_j = e^{\nu_j \tau}$, where $0 \leq \text{Im } \nu_j < 2\pi/\tau$. Then

$$\Lambda = \bigcup_{j=1}^m (\nu_j + \omega\mathbb{Z}). \quad (5.16)$$

First suppose that $\mu \notin \Lambda$. Then $H_\rho = ST$, where

$$S(\mu) = -(-D^2 + I)^{-1}(D + \mu I)^2, \quad T(\mu) = I - (D + \mu I)^{-2}U.$$

Since $P_N D = D P_N$, by (5.12) we have

$$\det H_\rho = \det S \det T.$$

The finite-dimensional approximation (5.12) of the determinant

$$\det T(\mu) = f(\mu) = \lim_{N \rightarrow \infty} \det(P_N T P_N^*),$$

converges absolutely for $\mu \notin \Lambda$ since

$$\operatorname{tr} |(D + \mu I)^{-2}U| < \infty.$$

Hence f is a holomorphic function on $\mathbb{C} \setminus \Lambda$ having at points in Λ poles of multiplicity not greater than double the multiplicity of the corresponding points of the spectrum of D .

The function f is periodic: $f(\mu + \omega) \equiv f(\mu)$. Indeed, if $\xi \in X^\mathbb{C}$, then $e^{\omega t} \xi \in X^\mathbb{C}$ and

$$(I - (D + \mu I)^{-2}U)e^{\omega t} \xi = e^{\omega t} (I - (D + (\mu + \omega)I)^{-2}U) \xi,$$

so $T(\mu)$ and $T(\mu + \omega)$ are similar. Thus $f(\mu) = \phi(e^{\mu\tau})$, where $\phi(\rho)$ is a meromorphic function having poles at the roots ρ_1, \dots, ρ_m of $\det(\rho I - Q)$. The multiplicity of the pole is at most twice the multiplicity of the corresponding root.

Hence there exists a polynomial $g(\rho)$ of degree $\leq 2m - 1$ such that the functions $\phi(\rho)$ and $g(\rho) \det^{-2}(\rho I - Q)$ have the same principal parts of the Laurent expansion at each pole. Since $\phi(\rho) \rightarrow 1$ as $|\rho| \rightarrow +\infty$, by Liouville's theorem,

$$\phi(\rho) = 1 + g(\rho) \det^{-2}(\rho I - Q). \quad (5.17)$$

The determinant $\det S$ converges conditionally. By (5.12),

$$\begin{aligned} \det(-(-D^2 + I)^{-1}(D + \mu I)^2) &= \lim_{N \rightarrow \infty} \prod_{\nu \in \Lambda_N} \frac{(\nu + \mu)^2}{\nu^2 - 1} \\ &= \lim_{N \rightarrow \infty} (-\mu^2)^k \prod_{\nu \in \Lambda_N, i\nu > 0} \left(\frac{\nu^2 - \mu^2}{\nu^2 - 1} \right)^2 = (-1)^k \lim_{N \rightarrow \infty} \prod_{\nu \in \Lambda_N} \frac{\nu^2 - \mu^2}{\nu^2 - 1}, \end{aligned}$$

where k is the multiplicity of zero in the spectrum of D . We have used that $\Lambda = -\Lambda$.

From (5.16) it follows that the last product converges absolutely. Hence it is a holomorphic function of $\rho_1, \dots, \rho_m, \rho$ for $\rho_j \neq e^{\pm\tau}$ and $\rho \neq 0$. To compute the product, we will use Euler's formula (see, for example, [30]):

$$\prod_{n \in \mathbb{Z}} \left(1 - \frac{\mu^2}{(\nu + \omega n)^2} \right) = \frac{\cosh \mu\tau - \cosh \nu\tau}{1 - \cosh \nu\tau}, \quad \nu \notin \omega\mathbb{Z}.$$

Suppose first that $\rho_j \neq 1$ and $\rho_i \neq \rho_j$ for $i \neq j$. Equivalently, $\nu_j \notin \omega\mathbb{Z}$ and $\nu_i - \nu_j \notin \omega\mathbb{Z}$ for $i \neq j$. Then by (5.16),

$$\begin{aligned} \prod_{\nu \in \Lambda} \frac{\nu^2 - \mu^2}{\nu^2 - 1} &= \prod_{\nu \in \Lambda} \left(1 - \frac{\mu^2}{\nu^2} \right) \left(1 - \frac{1}{\nu^2} \right)^{-1} \\ &= \prod_{j=1}^m \prod_{n \in \mathbb{Z}} \left(1 - \frac{\mu^2}{(\nu_j + \omega n)^2} \right) \left(1 - \frac{1}{(\nu_j + \omega n)^2} \right)^{-1} \\ &= \prod_{j=1}^m \frac{\cosh \mu\tau - \cosh \nu_j\tau}{\cosh \tau - \cosh \nu_j\tau} = \prod_{j=1}^m \frac{\rho + \rho^{-1} - \rho_j - \rho_j^{-1}}{e^\tau + e^{-\tau} - \rho_j - \rho_j^{-1}} \\ &= \prod_{j=1}^m \frac{e^\tau (\rho - \rho_j)^2}{\rho (e^\tau - \rho_j)^2} = \frac{e^{m\tau} \det^2(\rho I - Q)}{\rho^m \det^2(e^\tau I - Q)}. \end{aligned}$$

By continuity this holds for any $\rho_1, \dots, \rho_m \neq e^{\pm\tau}$ and $\rho \neq 0$. Hence

$$\det S = (-1)^k \frac{e^{m\tau} \det^2(\rho I - Q)}{\rho^m \det^2(e^\tau I - Q)}.$$

By (5.12) and (5.17),

$$\rho^m \det H_\rho = (-1)^k \beta^{-1} (\det^2(\rho I - Q) + g(\rho)). \quad (5.18)$$

Thus $G(\rho) = \rho^m \det H_\rho$ is a polynomial of degree $2m$ in ρ with leading coefficient $(-1)^k \beta^{-1}$.

We claim that the polynomials $G(\rho)$ and $F(\rho) = \det(P - \rho I)$ have the same roots. It is sufficient to prove that if $F(\rho) = 0$, then $G(\rho) = 0$ and the root has at least the same multiplicity.

If $F(\rho) = 0$, there exists a non-zero τ -periodic vector field $\xi \in X$ such that $(-D^2 + U)e^{\mu t}\xi(t) = 0$. Thus $H_\rho \xi = 0$. Suppose first that ρ is not an eigenvalue of Q , that is, $\mu \notin \Lambda$. Then $H_\rho = ST$, where S is invertible and $\text{tr}|T - I| < \infty$. Hence $T\xi = 0$ implies $\det T = 0$ (see, for example, [31]). Then $\det H_\rho = \det S \det T = 0$, and so $G(\rho) = 0$.

If $\mu \in \Lambda$, we can repeat the same argument replacing S and T , for instance, by

$$\tilde{S} = (-D^2 + I)^{-1}(-(D + \mu I)^2 + I), \quad \tilde{T} = (-(D + \mu I)^2 + I)^{-1}(-(D + \mu I)^2 + U).$$

We have proved that

$$G(\rho) = (-1)^k \beta^{-1} F(\rho).$$

It remains to show that $(-1)^k = \sigma(-1)^m$. Indeed, k is the dimension of the subspace on which the orthogonal operator Q is the identity, while $\sigma = (-1)^n$, where n is the dimension of the subspace on which Q is a reflection. Since the dimension $m - k - n$ of the complementary subspace is even, (5.13) is proved.

6 Degeneracy in Hill's formula

In this section we consider the case when the periodic orbit γ is degenerate, that is, the variational system has a non-zero τ -periodic solution ζ . Equivalently, the linear Poincaré map P has multiplier 1. Usually, this happens if the Lagrangian system has a time periodic first integral \mathcal{J} which is nondegenerate on γ . Then, as proved by Poincaré, the variational system has a non-zero periodic solution and a non-trivial linear time periodic first integral which is the linearization of \mathcal{J} at γ . Here are two standard examples.

1. Autonomous Lagrangian system. Then (M, \mathcal{L}) has the energy integral

$$\mathcal{H}(x, \dot{x}) = \langle p, \dot{x} \rangle - \mathcal{L}(x, \dot{x}), \quad p = \mathcal{L}_x(x, \dot{x}).$$

The variational system of a periodic orbit γ has a periodic solution $\zeta(t) = \dot{\gamma}(t)$. A particular case is a closed geodesic in a Riemannian metric.

2. A Lagrangian system with symmetry. Suppose the Lagrangian system (M, \mathcal{L}) admits a symmetry group $\psi_s: M \rightarrow M$, $s \in \mathbb{R}$, preserving \mathcal{L} . Let $\mathbf{w}(x) = \left. \frac{d}{ds} \right|_{s=0} \psi_s(x)$ be the corresponding symmetry field. Then

$$\mathcal{J}(x, \dot{x}, t) = \langle p, \mathbf{w}(x) \rangle, \quad p = \mathcal{L}_x(x, \dot{x}, t),$$

is the Noether first integral. The variational system of a periodic orbit γ has a τ -periodic solution $\zeta(t) = \mathbf{w}(\gamma(t))$. Here is one concrete example.

Planar 3-body problem. Here

$$\mathcal{L}(x, \dot{x}) = \frac{1}{2} \sum_{i=1}^3 m_i |\dot{x}_i|^2 + \sum_{i \neq j} \frac{m_i m_j}{|x_i - x_j|}, \quad x_i \in \mathbb{R}^2.$$

Fix the centre of mass at the origin, so that

$$M = \left\{ x = (x_1, x_2, x_3) \in (\mathbb{R}^2)^3 : \sum_{i=1}^3 m_i x_i = 0, x_i \neq x_j \right\}.$$

Rotations of \mathbb{R}^2 preserve \mathcal{L} and the corresponding symmetry field is $\mathbf{w}(x) = (Jx_1, Jx_2, Jx_3)$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The Noether integral is the angular momentum

$$\mathcal{J}(x, \dot{x}) = \sum_{i=1}^3 \langle Jx_i, m_i \dot{x}_i \rangle.$$

Since the system is autonomous, we have double degeneracy of any periodic orbit on which \mathcal{H} and \mathcal{J} are independent: 1 is an eigenvalue of P with multiplicity at least 4. Stationary periodic solutions (orbits of the symmetry group) have lower-order degeneracy.

In several recent years many periodic solutions for the 3-body problem have been found by variational methods [13]. However, we see that the ordinary Hill formula is degenerate for them. In this section we put forward an approach to this problem.

3. General Hamiltonian commutative symmetry. These examples are particular cases of Hamiltonian symmetries. Let us look at the system (M, \mathcal{L}) from the Hamiltonian point of view. Let \mathcal{H} be the Hamiltonian. Suppose that the system admits an algebra \mathfrak{g} of Hamiltonian symmetry fields \mathbf{v} generated by integrals $\mathcal{J}_{\mathbf{v}}$. Let γ be a τ -periodic solution in the phase space and P the corresponding monodromy operator. For any $\mathbf{v} \in \mathfrak{g}$, $\zeta(t) = \mathbf{v}(\gamma(t))$ is a periodic solution of the variational system. Therefore

$$Pw = w, \quad w = \mathbf{v}(\gamma(0)), \quad \mathbf{v} \in \mathfrak{g}. \quad (6.1)$$

If the system is autonomous, the Hamiltonian vector field of the system (M, \mathcal{L}) will be in \mathfrak{g} , and the corresponding eigenvector w is $\dot{\gamma}(0)$.

In the present paper we consider only the case when the k -dimensional algebra \mathfrak{g} is commutative. Then the corresponding eigenspace $V \subset \text{Ker}(P - I)$ is isotropic, and the multiplicity of eigenvalue 1 is at least $2k$. Hamiltonian reduction makes it possible to remove this degeneracy, but then the reduced system loses the natural Lagrangian structure.

The classical way to remove autonomous degeneracy is to pass from the Hamilton action functional to the Maupertuis action functional on the energy level [32]. The classical way to remove symmetry degeneracy in a Lagrangian system is the Routh method [32]. We briefly describe it here.

Suppose the system (M, \mathcal{L}) admits k commuting independent symmetry fields $\mathbf{w}^1, \dots, \mathbf{w}^k$ on M :

$$[\mathbf{w}^\alpha, \mathbf{w}^\beta] = 0, \quad \alpha, \beta = 1, \dots, k.$$

The corresponding flows of symmetry ψ_s^α commute. Let G be the (local) commutative group acting on M by $x \mapsto \psi_{s_1}^1 \circ \dots \circ \psi_{s_k}^k(x)$, $s \in \mathbb{R}^k$. Suppose that $\widetilde{M} = M/G$ is a smooth manifold and $\pi: M \rightarrow \widetilde{M}$ a smooth fibration.

The Noether integrals $\mathcal{J}^\alpha = \langle p, \mathbf{w}^\alpha \rangle$ give a vector integral $\mathcal{J}(x, \dot{x}, t) \in \mathbb{R}^k$. The Routh method reduces the Lagrangian system (M, \mathcal{L}) with fixed value $\mathcal{J} = c \in \mathbb{R}^k$ of the Noether integral to a Lagrangian system $(\widetilde{M}, \widetilde{\mathcal{L}})$ on the reduced configuration space \widetilde{M} .

For $c = 0$ the reduced Lagrangian is defined by¹⁰

$$\widetilde{\mathcal{L}}(x, \dot{x}, t) = \min_{s \in \mathbb{R}^k} \mathcal{L}(x, \dot{x} + s_\alpha \mathbf{w}^\alpha(x), t), \quad (6.2)$$

provided the minimum exists, for example, \mathcal{L} is superlinear in velocity. Since $\widetilde{\mathcal{L}}$ depends only on $\tilde{x} = \pi(x)$ and $\tilde{\dot{x}} = d\pi(x)\dot{x}$, it can be regarded as a function on $T\widetilde{M} \times \mathbb{R}$.

For $c \neq 0$ take closed G -invariant 1-forms ν_1, \dots, ν_k on M such that $\nu_\alpha(\mathbf{w}^\beta) \equiv \delta_\alpha^\beta$.¹¹ Then if we replace the Lagrangian by gauge-equivalent

$$\widehat{\mathcal{L}}(x, \dot{x}, t) = \mathcal{L}(x, \dot{x}, t) - c^\alpha \nu_\alpha(\dot{x}),$$

Lagrange's equations do not change, but the Noether integrals will be replaced by $\widehat{\mathcal{J}}^\alpha = \mathcal{J}^\alpha - c^\alpha$. Hence the value c of the Noether integral is replaced by 0 and so the Routh function $\widetilde{\mathcal{L}}$ can be defined by (6.2). The following theorem folds (see [32]).

Theorem 6.1 (Routh). *Let $x(t)$ be a trajectory of the system (M, \mathcal{L}) with $\mathcal{J} = c$. Then $\tilde{x}(t) = \pi(x(t))$ is a trajectory of the system $(\widetilde{M}, \widetilde{\mathcal{L}})$. Conversely, if $\tilde{x}(t)$ is a trajectory of the system $(\widetilde{M}, \widetilde{\mathcal{L}})$, then there exists a trajectory $x(t)$ of the system (M, \mathcal{L}) with $\mathcal{J} = c$ such that $\tilde{x}(t) = \pi(x(t))$.*

If γ is a periodic orbit of the system (M, \mathcal{L}) and $\tilde{\gamma}$ the corresponding orbit of the system $(\widetilde{M}, \widetilde{\mathcal{L}})$, then their variational systems are related by a linear version of Routh's method. In the next section we describe the Routh reduction for a linear Lagrangian system. It applies in a more general case, for example, when the Lagrangian system has non-Noether integrals. In particular, the linear Routh reduction includes the linearized Maupertuis reduction on an energy level.

¹⁰Recall that we assume summation in repeated Greek indices.

¹¹Such ν_α exists globally if the fibration $\pi: M \rightarrow \widetilde{M}$ is trivial. In general the first Chern class provides an obstruction. However ν_α always exists in a neighbourhood of a periodic orbit.

6.1 Routh reduction in a linear Lagrangian system

If the linear Poincaré map $P: W \rightarrow W$ of the periodic orbit γ has eigenvalue 1, then to any eigenvector $w = Pw$ there corresponds a non-zero τ -periodic solution $\zeta(t)$ of the variational system (E, Λ) . As proved by Poincaré, the variational system has a linear τ -periodic first integral

$$I_\zeta(\xi, D\xi) = (\zeta, D\xi) - (\xi, D\zeta).$$

Indeed, by (5.4)

$$\frac{d}{dt} I_\zeta(\xi(t), D\xi(t)) = (\zeta, D^2\xi) - (\xi, D^2\zeta) = (\zeta, U\xi) - (\xi, U\zeta) = 0.$$

In fact, $I_\zeta(\xi, D\xi) = \omega(w, v) = J_w(v)$ is the value of the symplectic form on the vectors $v, w \in W$ corresponding to ξ, ζ .

Suppose the Poincaré map P has several eigenvectors corresponding to unit eigenvalue. Let $V \subset \text{Ker}(P - I)$ be an isotropic subspace and let $\Gamma \subset X$ be the corresponding vector space of periodic solutions of the variational system (E, Λ) . Let w^1, \dots, w^k be a basis in V and $\zeta^1, \dots, \zeta^k \in \Gamma$ the corresponding independent solutions. The variational system has first integrals

$$I^\alpha(\xi, D\xi) = (\zeta^\alpha, D\xi) - (\xi, D\zeta^\alpha), \quad \alpha = 1, \dots, k,$$

in involution

$$I^\alpha(\zeta^\beta, D\zeta^\beta) = (\zeta^\alpha, D\zeta^\beta) - (\zeta^\beta, D\zeta^\alpha) = \omega(w^\alpha, w^\beta) = 0. \quad (6.3)$$

We write shortly $I = (I^1, \dots, I^k)$.

Denote

$$F_t = \{\zeta(t) : \zeta \in \Gamma\} = \text{span}\{\zeta^1(t), \dots, \zeta^k(t)\}.$$

To simplify the presentation we use the following non-degeneracy assumption.

Condition A'. $\dim F_t = k$ for all t .

Equivalently, $\zeta^1(t), \dots, \zeta^k(t) \in E_t$ are independent for all t . Thus the Gram matrix

$$G = (g^{\alpha\beta}), \quad g^{\alpha\beta}(t) = (\zeta^\alpha(t), \zeta^\beta(t)), \quad (6.4)$$

is nondegenerate for all t .

In Appendix A.3 we will show that this assumption is unnecessary. In fact, the set $\Sigma = \{t \in \mathbb{R}/\tau\mathbb{Z} : \dim F_t < k\}$ is finite and the family $(F_t)_{t \notin \Sigma}$ can be extended to a smooth k -dimensional vector bundle $(F_t)_{t \in \mathbb{R}/\mathbb{Z}}$. We will show that everything in this section works without the non-degeneracy assumption A'.

We describe Routh reduction for the linear system (E, Λ) . The reduced configuration spaces $\tilde{E}_t = E_t/F_t$ can be identified with

$$E_t^\perp = \{u \in E_t : (u, w) = 0 \text{ for all } w \in F_t\}$$

via the orthogonal projection $\Pi = \Pi_t: E_t \rightarrow E_t^\perp$. For a smooth field $\xi(t) \in E_t$ denote $D^\perp \xi(t) = \Pi_t D\xi(t)$. Explicitly,

$$\Pi \xi = \xi - g_{\alpha\beta}(\xi, \zeta^\beta) \zeta^\alpha, \quad D^\perp \xi = D\xi - g_{\alpha\beta}(D\xi, \zeta^\beta) \zeta^\alpha, \quad (6.5)$$

where $G^{-1} = (g_{\alpha\beta})$ is the inverse of the Gram matrix $G = (g^{\alpha\beta})$. In Appendix A.3 we show that Π and D^\perp are smooth also when the non-degeneracy assumption fails.

Define the Routh Lagrangian Λ^\perp on $E^\perp = (E_t^\perp)$ by

$$\Lambda^\perp(\eta, D^\perp \eta) = \frac{1}{2} (D^\perp \eta, D^\perp \eta) + \frac{1}{2} (U^\perp \eta, \eta), \quad \eta(t), D^\perp \eta(t) \in E_t^\perp,$$

where the symmetric operator $U^\perp(t): E_t^\perp \rightarrow E_t^\perp$ is given by

$$(U^\perp u, u) = (Uu, u) - 3g_{\alpha\beta}(u, D^\perp \zeta^\alpha)(u, D^\perp \zeta^\beta), \quad u \in E_t^\perp.$$

Thus, $U^\perp = \Pi U - 3C$, where

$$Cu = g_{\alpha\beta}(u, D^\perp \zeta^\alpha) D^\perp \zeta^\beta$$

(C is independent on the choice of the basis).

The bilinear action form of the system (E^\perp, Λ^\perp) is

$$\frac{1}{2} h^\perp(\eta, \eta) = \int_0^\tau \Lambda^\perp(\eta, D^\perp \eta) dt, \quad \eta(t) \in E_t^\perp. \quad (6.6)$$

We have Routh's theorem for linear Lagrangian systems.

Theorem 6.2. *Let $\xi(t) \in E_t$ be a solution of the system (E, Λ) such that $I(\xi, D\xi) \equiv 0$. Then $\eta(t) = \Pi \xi(t) \in E_t^\perp$ is a solution of the system (E^\perp, Λ^\perp) . Conversely, if $\eta(t)$ is a solution of the system (E^\perp, Λ^\perp) , then there exists a solution $\xi(t)$ of the system (E, Λ) , defined mod Γ , such that $I(\xi, D\xi) = 0$ and $\eta(t) = \Pi \xi(t)$.*

For the proof we need the following evident result.

Lemma 6.1. *Let*

$$\eta(t) - \xi(t) \in F_t, \quad \xi(t) = \eta(t) + \lambda_\alpha(t) \zeta^\alpha(t). \quad (6.7)$$

Then ξ satisfies $I_i^\alpha(\xi, D\xi) = c^\alpha$ for all $\alpha = 1, \dots, k$ if and only if

$$\dot{\lambda}_\alpha = g_{\alpha\beta}(c^\beta - I^\beta(\eta, D\eta)). \quad (6.8)$$

Indeed, $I^\alpha(\xi, D\xi) = I^\alpha(\eta, D\eta) + g^{\alpha\beta}\dot{\lambda}_\beta$.

Proof of Theorem 6.2. A vector field $\xi(t)$, $0 \leq t \leq \tau$, is a solution of (E, Λ) if and only if

$$h(\xi, \phi) = \int_0^\tau ((D\xi, D\phi) + (U\xi, \phi)) dt = 0$$

for any smooth variation $\phi(t) \in E_t$ such that $\phi(0) = \phi(\tau) = 0$.

Suppose $I(\xi, D\xi) = 0$ and let $\eta = \Pi\xi$. We need to show that for every smooth variation $\phi(t) \in E_t^\perp$ such that $\phi(0) = \phi(\tau) = 0$ we have

$$h^\perp(\eta, \phi) = \int_0^\tau ((D^\perp\eta, D^\perp\phi) + (U^\perp\eta, \phi)) dt = 0,$$

where h^\perp is the bilinear form (6.6) corresponding to the Routh system.

By Lemma 6.1, $\xi = \eta + \lambda_\alpha \zeta^\alpha$, where $\eta(t) \in E_t^\perp$ and

$$\dot{\lambda}_\alpha = -g_{\alpha\beta}I^\beta(\eta, D\eta) = 2g_{\alpha\beta}(\eta, D\zeta^\beta). \quad (6.9)$$

Since $(\eta, \zeta^\alpha) = (\phi, \zeta^\alpha) = 0$, (6.5) gives

$$D\eta = D^\perp\eta - g_{\alpha\beta}(\eta, D\zeta^\alpha)\zeta^\beta, \quad D\phi = D^\perp\phi - g_{\alpha\beta}(\phi, D\zeta^\alpha)\zeta^\beta.$$

We obtain

$$\begin{aligned} h(\xi, \phi) &= \int_0^\tau ((D\eta + \dot{\lambda}_\alpha \zeta^\alpha + \lambda_\alpha D\zeta^\alpha, D\phi) + (U\eta, \phi) + (\lambda_\alpha U\zeta^\alpha, \phi)) dt \\ &= \int_0^\tau ((D\eta, D\phi) + (U\eta, \phi) + (\dot{\lambda}_\alpha \zeta^\alpha + \lambda_\alpha D\zeta^\alpha, D\phi) + (\lambda_\alpha D^2\zeta^\alpha, \phi)) dt \\ &= \int_0^\tau \left((D^\perp\eta, D^\perp\phi) + (U\eta, \phi) + (g_{\alpha\beta}(\eta, D\zeta^\alpha)\zeta^\beta, g_{\delta\epsilon}(\phi, D\zeta^\delta)\zeta^\epsilon) \right. \\ &\quad \left. + \frac{d}{dt}((\lambda_\alpha D\zeta^\alpha, \phi) - 2(\dot{\lambda}_\alpha \zeta^\alpha, \phi)) \right) dt \\ &= \int_0^\tau ((D^\perp\eta, D^\perp\phi) + (U\eta, \phi) - 3g_{\alpha\beta}(\phi, D\zeta^\alpha)(\eta, D\zeta^\beta)) dt \\ &= h^\perp(\eta, \phi) = 0. \end{aligned}$$

Hence η is a trajectory of (E^\perp, Λ^\perp) . We skip the proof of the converse. \square

Now we can write Hill's formula for the reduced linear Poincaré map $\widetilde{P}: \widetilde{W} \rightarrow \widetilde{W}$. Let

$$X^\perp = \{\eta \in X : \eta(t) \in E_t^\perp\}$$

and let

$$H^\perp = (-D^{\perp 2} + I)^{-1}(-D^{\perp 2} + U^\perp): X^\perp \rightarrow X^\perp$$

be the Hessian operator for the reduced system (E^\perp, Λ^\perp) . Let $Q^\perp: E_0^\perp \rightarrow E_0^\perp$ be the operator of parallel transport corresponding to $D^\perp \eta = 0$ and let $\sigma^\perp = \det Q^\perp = \pm 1$. If assumption A' holds, then $\sigma = \sigma^\perp$ because the bundle F is oriented. In general F can be non-oriented.

Corollary 6.1.

$$\det(\tilde{P} - I) = \sigma^\perp (-1)^{m-k} \beta^\perp \det H^\perp, \quad \beta^\perp = e^{(m-k)\tau} \det^{-2}(e^\tau I - Q^\perp) > 0. \quad (6.10)$$

For the geodesic problem $Q^\perp = Q|_{X^\perp}$, $H^\perp = H|_{X^\perp}$, $\tilde{P} = P^\perp$, $\sigma = \sigma^\perp$, and we obtain Corollary 5.1.

Note that in general $H^\perp \neq H|_{X^\perp}$, except when $D^\perp \zeta = 0$. The reason is that if $\eta \in X^\perp$ is τ -periodic, λ in (6.9) is not periodic in general, and so $\xi \notin X$. Hence the space X^\perp of periodic $\eta(t) \in E_t^\perp$ does not correspond to the space of periodic $\xi(t) \in E_t$ such that $I(\xi, D\xi) = 0$. Thus h^\perp is not the restriction of h to X^\perp as in the geodesic case. Hence we need to discuss the relation between h^\perp and h .

6.2 Elimination of degeneracy in the action functional

As in (3.21), define subspaces $Y, Z \subset X$ by

$$Y = \{\xi \in X : I(\xi, D\xi) \equiv \text{const}\}, \quad (6.11)$$

$$Z = \{\xi \in X : \xi(t) \in F_t \text{ for all } t\}. \quad (6.12)$$

If $\zeta^1, \dots, \zeta^k \in \Gamma$ are basis periodic solutions, then

$$Z = \left\{ \xi(t) = \lambda_\alpha(t) \zeta^\alpha(t) : \lambda_\alpha(t + \tau) = \lambda_\alpha(t), \int_0^\tau g^{\alpha\beta} \dot{\lambda}_\alpha \dot{\lambda}_\beta dt < \infty \right\}.$$

Lemma 6.2. *For any $\eta \in X$ there exists $\xi = \Phi(\eta) \in Y$, unique mod Γ , such that $\xi - \eta \in Z$. Explicitly, $\eta = \xi - \lambda_\alpha \zeta^\alpha$, where λ_α satisfies (6.8) with $c^\alpha = c^\alpha(\eta)$ given by*

$$c^\alpha = \kappa^{\alpha\beta} \int_0^\tau g_{\beta\delta} I^\delta(\eta, D\eta) dt, \quad (\kappa^{\alpha\beta}) = (\bar{g}_{\alpha\beta})^{-1}, \quad \bar{g}_{\alpha\beta} = \int_0^\tau g_{\alpha\beta} dt. \quad (6.13)$$

This follows from Lemma 6.1 for periodic ξ and η . We have defined a projection $\Phi: X \rightarrow \hat{Y} = Y/\Gamma$ which is identical on Y and $\Phi = 0$ on Z . We obtain

Proposition 6.1. *$Y \cap Z = \Gamma$ and $Y + Z = X$.*

Proposition 6.2. *The spaces Y and Z are h -orthogonal. that is, $h(\xi, \eta) = 0$ for all $\xi \in Z$ and $\eta \in Y$. The restriction of h to Z has the form*

$$h(\lambda_\alpha \zeta^\alpha, \lambda_\beta \zeta^\beta) = \int_0^\tau g^{\alpha\beta} \dot{\lambda}_\alpha \dot{\lambda}_\beta dt. \quad (6.14)$$

Proof. Take $\xi \in Z$, $\xi(t) = \lambda_\alpha(t) \zeta^\alpha(t)$. Then

$$\begin{aligned} h(\lambda_\alpha \zeta^\alpha, \eta) &= \int_0^\tau \left((D(\lambda_\alpha \zeta^\alpha), D\eta) + (U\lambda_\alpha \zeta^\alpha, \eta) \right) dt \\ &= \int_0^\tau \left((\dot{\lambda}_\alpha \zeta^\alpha, D\eta) + (\lambda_\alpha D\zeta^\alpha, D\eta) + (\lambda_\alpha D^2 \zeta^\alpha, \eta) \right) dt \\ &= \int_0^\tau \left(\frac{d}{dt}(\eta, \lambda_\alpha D\zeta^\alpha) + \dot{\lambda}_\alpha I^\alpha(\eta, D\eta) \right) dt \\ &= (\eta, \lambda_\alpha D\zeta^\alpha)|_0^\tau + \int_0^\tau \dot{\lambda}_\alpha I^\alpha(\eta, D\eta) dt \end{aligned} \quad (6.15)$$

(we have used that ζ satisfies the variational system). If $\eta = \lambda_\alpha \zeta^\alpha$, then $I^\alpha(\eta, D\eta) = g^{\alpha\beta} \dot{\lambda}_\alpha \dot{\lambda}_\beta$. Hence

$$h(\lambda_\alpha \zeta^\alpha, \lambda_\beta \zeta^\beta) = \int_0^\tau g^{\alpha\beta} \dot{\lambda}_\alpha \dot{\lambda}_\beta dt + (\lambda_\alpha \zeta^\alpha, \lambda_\beta D\zeta^\beta)|_0^\tau. \quad (6.16)$$

If $\eta \in Y$ and the λ_α are periodic, (6.15) gives 0 and (6.16) gives (6.14), which proves Proposition 6.2. \square

Let

$$\widehat{X} = X/\Gamma, \quad \widehat{Y} = Y/\Gamma, \quad \widehat{Z} = Z/\Gamma.$$

Then $\widehat{Y} \oplus \widehat{Z} = \widehat{X}$. The bilinear form h defined a form \hat{h} on \widehat{X} and $\widehat{Y} \perp_{\hat{h}} \widehat{Z}$, while $\hat{h}|_{\widehat{Z}}$ is positive definite.

Corollary 6.2.

$$\text{ind } h = \text{ind } \hat{h}|_{\widehat{Y}}, \quad \text{null } h = \text{null } \hat{h}|_{\widehat{Y}} + k.$$

Corollary 6.3. *The projection $\Pi: X \rightarrow X^\perp$ defines an isomorphism*

$$\widehat{\Pi}|_{\widehat{Y}}: \widehat{Y} \rightarrow X^\perp, \quad (\Pi|_{\widehat{Y}})^{-1} = \Phi|_{X^\perp}. \quad (6.17)$$

Indeed, if $\Pi\xi = 0$ for $\xi \in Y$, then $\xi \in Z$ and hence $\xi \in Y \cap Z = \Gamma$ by Proposition 6.1. Similarly, $Y + Z = X$ implies that $\Pi(Y) = X^\perp$.

Next we compute the restriction $h|_Y$. Let $h^\perp(\eta, \eta)$ be the bilinear form for the reduced system (E^\perp, Λ^\perp) .

Proposition 6.3. Let $h^\top = h|_Y \circ \widehat{\Pi}^{-1}$ be the bilinear form on X^\perp corresponding to $h|_Y$. Then for any $\eta \in X^\perp$,

$$h^\top(\eta, \eta) = h^\perp(\eta, \eta) + \bar{g}_{\alpha\beta} c^\alpha c^\beta, \quad (6.18)$$

where the $c^\alpha(\eta) = I^\alpha(\xi, D\xi)$, $\xi = \Phi\eta$, are defined by (6.13).

This follows from a more general formula.

Lemma 6.3. Let $\xi(t) = \eta(t) + \lambda_\alpha(t)\zeta^\alpha(t)$, $0 \leq t \leq \tau$, be as in Lemma 6.1. Then

$$\begin{aligned} h(\xi, \bar{\xi}) &= h(\eta, \eta) - \int_0^\tau g_{\alpha\beta} (I^\alpha(\eta, D\eta) - c^\alpha) (I^\beta(\eta, D\eta) - c^\beta) dt \\ &\quad + ((\lambda_\alpha \zeta^\alpha, \lambda_\beta D\zeta^\beta) + 2(\lambda_\alpha \eta, D\zeta^\alpha) + 2c^\alpha \lambda_\alpha)|_0^\tau. \end{aligned} \quad (6.19)$$

Proof. We have

$$h(\xi, \xi) = h(\eta, \eta) + 2h(\eta, \lambda_\alpha \zeta^\alpha) + h(\lambda_\alpha \zeta^\alpha, \lambda_\beta \zeta^\beta).$$

Now (6.19) follows from (6.15), (6.16), and (6.8). \square

Proof of Proposition 6.3. If η and λ are periodic, then the boundary terms in (6.19) vanish. By (6.13),

$$\begin{aligned} &\int_0^\tau g_{\alpha\beta} (c^\alpha - I^\alpha(\eta, D\eta)) (c^\beta - I^\beta(\eta, D\eta)) dt \\ &= \int_0^\tau g_{\alpha\beta} I^\alpha(\eta, D\eta) I^\beta(\eta, D\eta) dt - 2 \int_0^\tau g_{\alpha\beta} c^\alpha I^\beta(\eta, D\eta) dt + \bar{g}_{\alpha\beta} c^\alpha c^\beta \\ &= \int_0^\tau g_{\alpha\beta} I^\alpha(\eta, D\eta) I^\beta(\eta, D\eta) dt - \bar{g}_{\alpha\beta} c^\alpha c^\beta. \end{aligned}$$

Next we use $\eta = \Pi\xi \in X^\perp$. Then $I^\alpha(\eta, D\eta) = -2(\eta, D^\perp \zeta^\alpha)$, and so

$$\int_0^\tau g_{\alpha\beta} I^\alpha(\eta, D\eta) I^\beta(\eta, D\eta) dt = 4 \int_0^\tau g_{\alpha\beta} (\eta, D^\perp \zeta^\alpha) (\eta, D^\perp \zeta^\beta) dt.$$

Finally,

$$\begin{aligned} h(\eta, \eta) &= \int_0^\tau ((D^\perp \eta, D^\perp \bar{\eta}) + g_{\alpha\beta} (\eta, D^\perp \zeta^\alpha) (\eta, D^\perp \zeta^\beta) + (U\eta, \bar{\eta})) dt \\ &= h^\perp(\eta, \eta) + 4 \int_0^\tau g_{\alpha\beta} (\eta, D^\perp \zeta^\alpha) (\eta, D^\perp \zeta^\beta) dt. \end{aligned} \quad (6.20)$$

It remains to substitute (6.20) in (6.19). \square

From the point of view of Routh reduction it is natural to consider the space

$$Y^0 = \{\xi \in X : I(\xi, D\xi) \equiv 0\} \subset Y. \quad (6.21)$$

Indeed, by (6.18), $h|_{Y^0} = h^\perp \circ \Pi|_{Y^0}$.

Proposition 6.4.

$$Y^0 + Z = \left\{ \eta \in X : \int_0^\tau g_{\alpha\beta}(D^\perp \zeta^\beta, \eta) dt = 0 \right\}. \quad (6.22)$$

Proof. We take $\eta \in X$. Then $\xi = \Phi\eta \in Y^0$ provided that

$$0 = \int_0^\tau g_{\alpha\beta} I^\beta(\eta, D\eta) dt = -2 \int_0^\tau g_{\alpha\beta}(D^\perp \zeta^\beta, \eta) dt.$$

Here we have used that

$$g_{\alpha\beta} I^\beta(\eta, D\eta) = \frac{d}{dt}(g_{\alpha\beta}(\eta, \zeta^\beta)) - 2g_{\alpha\beta}(\eta, D^\perp \zeta^\beta). \quad (6.23)$$

□

We see that $Y^0 + Z = X$ if and only if $D^\perp \zeta^\alpha = 0$. Equivalently, $DZ \subset Z$. Let

$$X_0^\perp = \Pi Y^0 = \left\{ \eta \in X^\perp : \int_0^\tau g_{\alpha\beta}(D^\perp \zeta^\beta, \eta) dt = 0 \right\}. \quad (6.24)$$

Then X_0^\perp has codimension $\leq k$ in X^\perp . We have $h^\top \geq h^\perp$ and $h^\top = h^\perp$ on X_0^\perp . Since $\text{ind } h^\top = \text{ind } h$,

$$\text{ind } h^\perp \leq \text{ind } h|_X \leq \text{ind } h^\perp + k.$$

Let $\Omega = Y/Y^0$. Since $\widehat{\Pi}: Y \rightarrow X^\perp$ is an isomorphism, $\dim \Omega \leq k$. The integral $I: Y \rightarrow \mathbb{R}^k$ gives a map $\Omega \rightarrow \mathbb{R}^k$. To compare the indices of h^\top and h^\perp , in § 6.4 we construct a basis in Ω , on which I is nondegenerate.

6.3 Indices of h and h^\perp

In this section we discuss the relation between $\text{ind } h = \text{ind } h|_Y$ and $\text{ind } h^\perp$. Let $P: W \rightarrow W$ be the Poincaré map. As in § 3.7, we assume that

$$N = \text{Ker}(P - I)^2 = \{v \in W : Pv - v \in V\}$$

is symplectic and $V = \text{Ker}(P - I)$ is a Lagrangian subspace in N . Let w^1, \dots, w^k be a basis in V and q_1, \dots, q_k a basis in a Lagrangian complement to V in N . We define the matrix $s_{\alpha\beta}$ by formula (3.30) and the matrices $A = (a_{\alpha\beta})$ and $A^\perp = (a_{\alpha\beta}^\perp)$ by (3.31), where $\varkappa^{\alpha\beta}$ and $\bar{g}_{\alpha\beta}$ are the matrices in (6.13).

Definition 6.1. We say that γ is nondegenerate mod V if Assumption A' (p. 56) holds and $\det A^\perp \neq 0$.

Theorem 6.3. *Suppose that γ is nondegenerate mod V . Then*

$$\text{ind } h - \text{ind } h^\perp = \text{ind } A - \text{ind } A^\perp. \quad (6.25)$$

The formulation coincides with Theorem 3.2, but the proof is different. We prove Theorem 6.3 in § 6.4. Since $h|_{\widehat{Z}}$ is positive definite, as in the proof of Corollary 3.3, we obtain

Corollary 6.4. *Suppose $\det A \neq 0$. Then*

$$(-1)^{\text{ind } h} = (-1)^{\text{ind } h^\perp + \text{ind } b}.$$

6.4 The spaces Ω and Ω^\perp

Consider periodic solutions $\zeta^\alpha(t)$, $\eta_\alpha(t)$ of the system (E, Λ) which correspond to w^α , q_α . Then the $\zeta^\alpha(t)$ are periodic and satisfy (6.3). Equations (3.30) imply

$$\eta_\alpha(t + \tau) - \eta_\alpha(t) = s_{\alpha\beta} \zeta^\beta(t), \quad (\eta_\alpha, D\eta_\beta) - (\eta_\beta, D\eta_\alpha) = 0 \quad (6.26)$$

and $(\zeta^\alpha, D\eta_\beta) - (\eta_\beta, D\zeta^\alpha) = \delta_\beta^\alpha$.

For any $\alpha = 1, \dots, k$ we put

$$\hat{\eta}_\alpha = \eta_\alpha - \lambda_{\alpha\beta} \zeta^\beta,$$

where the coefficients $\lambda_{\alpha\beta}$ are chosen so that the $\hat{\eta}_\alpha$ are τ -periodic and

$$(\zeta^\alpha, D\hat{\eta}_\beta) - (\hat{\eta}_\beta, D\zeta^\alpha) = c_\beta^\alpha = \text{const}.$$

Then the c_β^α satisfy (3.38) and

$$\lambda_{\alpha\beta}(t + \tau) - \lambda_{\alpha\beta}(t) = s_{\alpha\beta}, \quad \dot{\lambda}_{\alpha\beta} = s_{\alpha\delta} \kappa^{\delta\varepsilon} g_{\varepsilon\beta}.$$

We define $\eta_\alpha^\perp = \Pi \eta_\alpha$. Then

$$\eta_\alpha^\perp = \eta_\alpha - \lambda_{\alpha\beta}^\perp \zeta^\beta, \quad \lambda_{\alpha\beta}^\perp(t + \tau) - \lambda_{\alpha\beta}^\perp(t) = s_{\alpha\beta}, \quad \lambda_{\alpha\beta}^\perp = (\eta_\alpha, \zeta^\delta) g_{\delta\beta}.$$

Consider the spaces

$$\Omega = \text{span}(\hat{\eta}_1, \dots, \hat{\eta}_k) = \Phi \Omega^\perp \subset \widehat{Y}, \quad \Omega^\perp = \text{span}(\eta_1^\perp, \dots, \eta_k^\perp) = \Pi \Sigma \subset X^\perp.$$

We also define $\widehat{\Omega}$ and \widehat{Y}^0 as the images of Ω and Y^0 under the canonical projection $\Pi_\Gamma: X \rightarrow \widehat{X} = X/\Gamma$.

Theorem 6.4. *Suppose $\det A^\perp \neq 0$. Then*

$$(a) \quad \widehat{Y} = \widehat{\Omega} \oplus \widehat{Y}^0, \quad X^\perp = \Omega^\perp \oplus \Pi Y^0;$$

(b) the maps $\hat{\Pi}|_{\hat{\Omega}}: \hat{\Omega} \rightarrow \Omega^\perp$ and $\hat{\Pi}|_{\hat{Y}^0}: \hat{Y}^0 \rightarrow X_0^\perp$ are linear isomorphisms;

(c) $h|_{Y^0} = h^\perp|_{X_0^\perp} \circ \Pi|_{Y^0}$ and $\hat{h}|_{\hat{Y}^0} = h^\perp|_{X_0^\perp} \circ \hat{\Pi}|_{\hat{Y}^0}$;

(d) for any $\xi \in Y^0$ and $\alpha, \beta = 1, \dots, k$,

$$h(\hat{\eta}_\alpha, \xi) = 0, \quad h(\hat{\eta}_\alpha, \hat{\eta}_\beta) = a_{\alpha\beta};$$

(e) for any $\xi^\perp \in \Pi Y^0$ and $\alpha, \beta = 1, \dots, k$

$$h^\perp(\eta_\alpha^\perp, \xi^\perp) = 0, \quad h^\perp(\eta_\alpha^\perp, \eta_\beta^\perp) = a_{\alpha\beta}^\perp.$$

The proof of Theorem 6.4 is contained in § A.2.

Corollary 6.5. *In the basis $\hat{\eta}_1, \dots, \hat{\eta}_k$*

$$\hat{h}|_{\hat{\Omega}} - h^\perp \circ \Pi|_{\hat{\Omega}} = (s_{\alpha\delta} \kappa^{\delta\varepsilon} s_{\varepsilon\beta} - 2s_{\alpha\beta} + \bar{g}_{\alpha\beta}) = SKS - 2S + \bar{G}.$$

This quadratic form is positive definite.

Indeed, let Q be the square root of K , that is, the positive definite symmetric matrix such that $Q^2 = K$. Then $Q^{-2} = \bar{G}$ and

$$SKS - 2S + \bar{G} = RR^*, \quad R = (S - \bar{G})Q.$$

This matrix is positive definite because R is nondegenerate.

Now we prove Theorem 6.3. Recall that by Proposition 6.2 and Theorem 6.4 we have the h -orthogonal expansion $X = Z \oplus \Omega \oplus Y^0$ and the h^\perp -orthogonal expansion $X^\perp = \Omega^\perp \oplus \Pi Y^0$. The form $h|_Z$ is positive definite and $h|_{Y^0} = h^\perp|_{X_0^\perp} \circ \Pi|_{Y^0}$. Therefore,

$$\text{ind } h - \text{ind } h^\perp = \text{ind } h|_{\Omega} - \text{ind } h^\perp|_{\Omega^\perp} = \text{ind } A - \text{ind } A^\perp.$$

6.5 Example: autonomous systems

Suppose the Lagrangian system is autonomous, so the variational system of a periodic trajectory γ has a periodic solution $\zeta(t) = \dot{\gamma}(t)$. If γ is nondegenerate in the autonomous sense (only two unit multipliers) then, as proved by Poincaré, there exists a family¹² γ_α of $\tau(\alpha)$ -periodic orbits such that $\gamma_0 = \gamma$ and $\tau(0) = \tau$ ([33], [34]). Let $E(\alpha) = \mathcal{H}|_{\gamma_\alpha}$ and $A(\alpha) = \int_{\gamma_\alpha} \langle p, dx \rangle$ be the energy and Maupertuis action of γ_α .

¹²It seems more natural to parametrize the family by the period τ . However, this is not always possible because it may happen that $\tau'(\alpha) = 0$.

Lemma 6.4. *Suppose that $dE/d\tau \neq 0$. Then*

$$(-1)^{\text{ind } b} = -\text{sign}(\tau'(\alpha)E'(\alpha)) = -\text{sign} \frac{dE}{d\tau}.$$

Proof. The union of trajectories of γ_α in the phase space $TM \cong T^*M$ is a symplectic cylinder Σ . Restricting the Hamiltonian system to Σ we obtain an integrable Hamiltonian system with one degree of freedom and Hamiltonian $H(\vartheta, I) = E(I)$, where $\vartheta \in \mathbb{R}/\mathbb{Z}$, $I \in \mathbb{R}$. Then we can assume $\alpha = I$, $\gamma_\alpha(t) = (E'(\alpha)t, \alpha)$. Then $\tau(\alpha) = 1/\nu(\alpha)$, where the frequency $\nu(\alpha)$ is $E'(\alpha)$. We have $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

The monodromy matrix of γ_α is $P = P_\alpha = \begin{pmatrix} 1 & \nu'/\nu \\ 0 & 1 \end{pmatrix}$. Thus $Pv = \begin{pmatrix} \nu'/\nu \\ 1 \end{pmatrix}$, and so $s = \omega(Pv, v) = \nu'/\nu = -\tau^{-2} d\tau/dE$. \square

As usual, we denote $\text{ind } \gamma = \text{ind } h(\gamma)$.

Proposition 6.5. *Let a periodic trajectory γ have exactly 2 unit multipliers. Suppose that $\sigma(-1)^{m+\text{ind } \gamma} dE/d\tau < 0$. Then γ has a real multiplier $\rho > 1$.*

Proof. Since $\sigma = \sigma^\perp$, by Corollary 6.4 and Lemma 6.4,

$$(-1)^{\text{ind } h^\perp} = -\text{sign}\left(\frac{dE}{d\tau}\right)(-1)^{\text{ind } \gamma}.$$

The dimension of the reduced system is $m^\perp = m - 1$. Hence

$$\sigma(-1)^{m^\perp + \text{ind } h^\perp} = \sigma(-1)^{m-1 + \text{ind } \gamma} \left(-\text{sign} \frac{dE}{d\tau}\right) = -1,$$

and by Corollary 5.2 applied to the reduced Hill formula (6.10), there exists a multiplier $\rho > 1$. \square

Example 6.1. Suppose a particle in \mathbb{R}^m moves under the potential field with homogeneous potential energy

$$V(\lambda x) = \lambda^k V(x), \quad \lambda > 0, \quad k(k-2) \neq 0.$$

Suppose γ is a τ -periodic solution with energy E . Then $\gamma_\lambda(t) = \lambda\gamma(\lambda^{k/2-1}t)$ is a periodic solution with period $\tau(\lambda) = \lambda^{1-k/2}\tau$ and energy $E(\lambda) = \lambda^k E$. Hence

$$\frac{dE(\lambda)}{d\tau(\lambda)} = \frac{2k}{k-2} \left(\frac{\tau(\lambda)}{\tau}\right)^{(k+2)/(k-2)} E. \quad (6.27)$$

Thus by Lemma 6.4

$$(-1)^{\text{ind } b} = \text{sign} \frac{2-k}{k}.$$

Consider the problem of the motion of a particle in \mathbb{R}^m in the force field generated by a homogeneous potential of degree k , where $k(k-2) \neq 0$. Equations $\sigma = 1$, (6.27), and Proposition 6.5 immediately imply

Proposition 6.6. *Let a periodic trajectory γ have exactly 2 unit multipliers. Suppose that $(-1)^{m+\text{ind } \gamma}(k-2)/k < 0$. Then γ has a real multiplier $\rho > 1$.*

6.6 Degeneracy in the ρ -index form

We have seen that the relation between $\text{ind } h$ and $\text{ind } h^\perp$ is not evident. This simplifies drastically for $\rho \neq 1$. Let X_ρ , $\rho \in S^1$, be the set of complex ρ -quasiperiodic vector fields. Similarly to (6.11), (6.12) define subspaces $Y_\rho, Z_\rho \subset X_\rho$:

$$Y_\rho = \{\xi \in X_\rho : I(\xi, D\xi) \equiv \text{const}\}, \quad Z_\rho = \{\xi \in X_\rho : \xi(t) \in F_t\}.$$

It is easy to see that for $\rho \neq 1$ and $\xi \in X_\rho$, $I(\xi, D\xi) \equiv c$ implies $c = 0$. Thus,

$$Y_\rho = Y_\rho^0 = \{\xi \in X_\rho : I(\xi, D\xi) \equiv 0\}.$$

Proposition 6.7. *For $\rho \neq 1$ we have $Y_\rho \cap Z_\rho = \{0\}$ and $X_\rho = Y_\rho \oplus Z_\rho$.*

Proof. We will define a projection $\Phi_\rho: X_\rho \rightarrow Y_\rho$ along Z_ρ . Take $\eta \in X_\rho$ and look for $\lambda_\alpha(t)$ such that $\xi = \eta + \lambda_\alpha \zeta^\alpha \in Y_\rho$. Then by (6.8), $\dot{\lambda}_\alpha = f_\alpha$, where $f_\alpha(t+\tau) = \rho f_\alpha(t)$. Hence $f_\alpha(t) = e^{\mu t} b_\alpha(t)$, where $\mu = \tau^{-1} \ln \rho$ and $b_\alpha(t)$ is a τ -periodic function:

$$b_\alpha(t) = \sum_{k \in \mathbb{Z}} b_{\alpha k} e^{k\omega t}, \quad \omega = \frac{2\pi i}{\tau}.$$

We obtain a unique solution $\lambda_\alpha(t)$ such that $\lambda_\alpha(t+\tau) = \rho \lambda_\alpha(t)$:

$$\lambda_\alpha(t) = e^{\mu t} \sum_{k \in \mathbb{Z}} \frac{b_{\alpha k} e^{k\omega t}}{\mu + k\omega}. \quad (6.28)$$

The denominator is non-zero if $\rho \neq 1$. □

Proposition 6.8. *The spaces Y_ρ, Z_ρ are h -orthogonal:*

$$h(\xi, \bar{\eta}) = 0 \quad \text{for all } \xi \in Z_\rho, \eta \in Y_\rho.$$

The restriction of h to Z_ρ is positive definite for $\rho \neq 1$:

$$h(\lambda_\alpha \zeta^\alpha, \bar{\lambda}_\beta \zeta^\beta) = \int_0^\tau g^{\alpha\beta} \dot{\lambda}_\alpha \dot{\bar{\lambda}}_\beta dt. \quad (6.29)$$

Proof. Take $\xi(t) = \lambda_\alpha(t)\zeta^\alpha(t) \in Z_\rho$ and $\eta \in Y_\rho$. Then by (6.15),

$$h(\lambda_\alpha \zeta^\alpha, \bar{\eta}) = (\lambda_\alpha D\zeta^\alpha, \bar{\eta})|_0^\tau = (|\rho|^2 - 1)(\lambda_\alpha(0)D\zeta^\alpha(0), D\eta(0)) = 0. \quad (6.30)$$

The proof of (6.29) is similar. \square

Next we compute the restriction of h to Y_ρ . Let

$$X_\rho^\perp = \{\eta \in X_\rho : \eta(t) \in E_t^\perp\},$$

and let $\Pi: X_\rho \rightarrow X_\rho^\perp$ be the projection (6.5). Since $X_\rho = Y_\rho \oplus Z_\rho = X_\rho^\perp \oplus Z_\rho$, $\Pi|_{Y_\rho}: Y_\rho \rightarrow X_\rho^\perp$ is an isomorphism and its inverse is $\Phi_\rho|_{X_\rho^\perp}$.

Proposition 6.9. *For $\rho \neq 1$ the bilinear form $h_\rho^\top = h \circ (\Pi|_{Y_\rho})^{-1}$ on X_ρ^\perp is equal to the Routh form h_ρ^\perp .*

This follows from Lemma 6.1 (for complex vector fields) since $c = 0$ and $\lambda(\tau) = \rho\lambda(0)$, $\eta(\tau) = \rho\eta(0)$, $|\rho| = 1$.

Corollary 6.6. *For $\rho \neq 1$ the ρ -index of the system (E, λ) equals the ρ -index of the Routh system (E^\perp, Λ^\perp) .*

Proposition 6.9 is not true for $\rho = 1$. Then the relation between indices is more complicated, as we saw before.

7 Reversible case

Suppose the Lagrangian system (M, \mathcal{L}) is reversible: there is an involution $S: M \rightarrow M$ which is a time reversing symmetry for \mathcal{L} :

$$\mathcal{L}(S(x), dS(x)\dot{x}, t) = \mathcal{L}(x, -\dot{x}, -t).$$

Let $\tau = 2T$. Then for any τ -periodic curve $\gamma \in \Omega$,

$$\mathcal{A}(\gamma) = \int_{-T}^T \mathcal{L}(\gamma(t), \dot{\gamma}(t), t) dt = \mathcal{A}(\tilde{\gamma}),$$

where $\tilde{\gamma}(t) = S\gamma(-t)$. Thus, the involution $R: \Omega \rightarrow \Omega$,

$$R(\gamma)(t) = \tilde{\gamma}(t) = S(\gamma(-t)),$$

preserves \mathcal{A} . A τ -periodic orbit γ is called reversible if $R(\gamma) = \gamma$. Then

$$S\gamma(-t) = \gamma(t), \quad S\gamma(T - t) = \gamma(T + t).$$

Hence $\gamma(0)$ and $\gamma(T)$ belong to the set N of fixed points of S . It is easy to see that γ is a reversible periodic orbit if and only if $\gamma_+ = \gamma|_{[0,T]}$ is a critical point of the action functional

$$\mathcal{A}_+(\nu) = \int_0^T \mathcal{L}(\nu(t), \dot{\nu}(t), t) dt$$

on the set Ω_+ of curves $\nu: [0, T] \rightarrow M$ with end-points in N .

Let $X = T_\gamma \Omega$ be the set of vector fields along γ and $J = dR(\gamma): X \rightarrow X$. Then

$$(J\xi)(t) = J_{-t}\xi(-t), \quad J_t = dS(\gamma(t)): E_t \rightarrow E_{-t}.$$

Since R preserves \mathcal{A} , the involution J preserves the Hessian bilinear form:

$$h(J\xi, J\eta) = h(\xi, \eta).$$

Since the operators D and U are intrinsically associated with h ,

$$J^* = J, \quad DJ = -JD, \quad UJ = JU.$$

Let $X_\pm = \{\xi \in X : J\xi = \pm\xi\}$. Then $X = X_+ \oplus X_-$ and any $\xi \in X$ is represented as $\xi = \xi_+ + \xi_-$, where $\xi_\pm \in X_\pm$. Then

$$h(\xi, \xi) = h(\xi_+, \xi_+) + h(\xi_-, \xi_-).$$

Since $DJ = -JD$, we have $D^2: X_\pm \rightarrow X_\pm$. Hence the Hessian operator $H = (-D^2 + I)^{-1}(-D^2 + U)$ commutes with J , and so $H: X_\pm \rightarrow X_\pm$. Denote $H_\pm = H|_{X_\pm}$.

Proposition 7.1. $H = H_+ \oplus H_-$ and $\det H = \det H_+ \det H_-$.

Next we give more explicit formulae for h_\pm . Any $\xi \in X_\pm$ is determined by the restriction

$$\xi|_{[0,T]} \in Y_\pm = \{\eta \in Y : \eta(0) \in E_0^\pm, \eta(T) \in E_T^\pm\}, \quad Y = W^{1,2}([0, T], E),$$

where

$$E_0^\pm = \{v \in E_0 : J_0 v = \pm v\}, \quad E_T^\pm = \{v \in E_T : J_T v = \pm v\}.$$

Thus we have the orthogonal decompositions

$$E_0 = E_0^+ \oplus E_0^-, \quad E_T = E_T^+ \oplus E_T^-.$$

For $\eta \in Y_\pm$ the corresponding $\xi \in X_\pm$ is given by

$$\xi|_{[0,T]} = \eta, \quad \xi|_{[-T,0]} = \pm J\eta.$$

Thus,

$$h(\xi, \xi) = h(\eta, \eta) + h(J\eta, J\eta) = 2K(\eta, \eta),$$

where

$$K(\eta, \eta) = \int_0^T ((D\eta, D\eta) + (U\eta, \eta)) dt$$

is the same form h , but considered on Y . Let $K_{\pm} = K|_{Y_{\pm}}$. Then $K_+ = d^2\mathcal{A}_+(\gamma_+)$ is the second variation of the functional \mathcal{A}_+ .

Let us consider the case $S = \text{id}$. Then $E_0^+ = E_0$, $E_T^+ = E_T$, $E_0^- = 0$, $E_T^- = 0$. Thus, $Y_+ = Y$ and

$$Y_- = Y_0 = \{\eta \in Y : \eta(0) = 0, \eta(T) = 0\}.$$

Corollary 7.1. *Let m be odd and $S = \text{id}$. If γ_+ is a nondegenerate minimum of \mathcal{A}_+ , then the corresponding reversible periodic orbit γ has a real multiplier > 1 .*

Indeed, γ preserves orientation, so $\sigma > 0$. The Hessian K_+ is positive definite, and hence the same is true for $K_- = K_+|_{Y_0}$.

An analogue of Corollary 7.1 is true also for m even.

Proposition 7.2. *If $S = \text{id}$ and γ_+ is a nondegenerate minimum of \mathcal{A}_+ , then h_{ρ} is positive definite for $|\rho| = 1$. Hence there are no multipliers on the unit circle.*

Proof. For complex $\xi \in X$, set

$$\eta(t) = e^{\mu t} \xi(t) = u(t) + iv(t), \quad u(t), v(t) \in E_t.$$

Then

$$h_{\rho}(\xi, \bar{\xi}) = h(\eta, \bar{\eta}) = h(u, u) + h(v, v).$$

Let us show that $h(u, u)$ is positive definite on $W^{1,2}([-T, T], E)$. Indeed, $u_+ = u|_{[0, T]} \in Y$ and $u_- = J(u|_{[-T, 0]}) \in Y$. Thus

$$h(u, u) = K(u_+, u_+) + K(u_-, u_-) > 0, \quad u \neq 0.$$

By Hill's formula $\det(P - \rho I) \neq 0$ for $\rho \in S^1$, and so P has no multipliers on S^1 . For another proof see [35]. \square

Consider again the case of a general involution S . Then $Y_+ \cap Y_- = Y_0$. Let Y_{\pm}^{\perp} be the K_{\pm} -complement of Y_0 in Y_{\pm} , that is, the set of $\eta \in Y_{\pm}$ such that $K_{\pm}(\eta, \zeta) = 0$ for all $\zeta \in Y_0$. By integration by parts,

$$Y_{\pm}^{\perp} = Y_{\pm} \cap Y^{\perp}, \quad Y^{\perp} = \{\eta \in Y : (-D^2 + U)\eta = 0\}.$$

The restriction of K_\pm to Y_\pm^\perp equals

$$K_\pm^\perp(\eta, \eta) = (D\eta, \eta)|_0^T, \quad \eta \in Y_\pm^\perp. \quad (7.1)$$

Let $K_0 = K|_{Y_0}$ and $K^\perp = K|_{Y^\perp}$. Then we have

$$H_\pm \cong K_0 \oplus K_\pm^\perp, \quad H \cong K_0 \oplus K_0 \oplus K_+^\perp \oplus K_-^\perp.$$

It follows that if $\det K_0 \neq 0$, then $\text{sign det } H = (-1)^{\text{ind } K_+^\perp + \text{ind } K_-^\perp} = (-1)^{\text{ind } K^\perp}$.

Proposition 7.3. *If the time moments 0 and T are non-conjugate, then*

$$(-1)^{\text{ind } \gamma} = \text{sign det } H = (-1)^{\text{ind } K^\perp}.$$

If the time moments 0 and T are non-conjugate, $\dim Y_+^\perp = 2n$ and $\dim Y_-^\perp = 2(m - n)$, where $n = \dim N$.

The quadratic form K_\pm^\perp has a simple meaning, the Hessian of the discrete Lagrangian (Hamilton action function) defined locally as

$$L(x, y) = \mathcal{A}_+(\nu),$$

where $\nu: [0, T] \rightarrow M$ is a trajectory joining x and y .

A Appendix

A.1 Proof of Theorem 3.3

(a) By (3.37), for any constant vector λ^β we have:

$$I^\alpha(\lambda^\beta \hat{q}_{\beta i}, \lambda^\beta \hat{q}_{\beta i+1}) = c_\beta^\alpha \lambda^\beta.$$

By condition C, the matrix c_α^β is nondegenerate. Therefore the equation

$$I^\alpha(\lambda^\beta \hat{q}_{\beta i}, \lambda^\beta \hat{q}_{\beta i+1}) = r^\alpha$$

with respect to λ^β is solvable for any constant vector r^α . This implies the first statement in (a).

To prove the second statement in (a), we show that $\dim \Omega^\perp = k$ and $\Omega^\perp \cap X_0^\perp = 0$. In view of equation (3.26), the last two conditions are equivalent to the non-degeneracy of the matrix

$$e_{\alpha\beta} = \langle \mathbf{d}_\alpha, \mathbf{q}_\beta^\perp \rangle = \sum_i g_{\alpha\delta i} I_i^\delta(q_{\beta i}^\perp, q_{\beta i+1}^\perp).$$

By (3.36) we have

$$\begin{aligned}
e_{\alpha\beta} &= \sum_i g_{\alpha\delta i} (-\langle B_i q_{\beta i}^\perp, w_{i+1}^\delta \rangle + \langle B_i w_i^\delta, q_{\beta i+1}^\perp \rangle) \\
&= \sum_i g_{\alpha\delta i} (\delta_\beta^\delta + \lambda_{\beta\epsilon i}^\perp g_i^{\epsilon\delta} - \lambda_{\beta\epsilon i+1}^\perp g_i^{\delta\epsilon}) \\
&= \bar{g}_{\alpha\beta} - \sum_i (\lambda_{\beta\epsilon i+1}^\perp - \lambda_{\beta\epsilon i}^\perp) = \bar{g}_{\alpha\beta} - s_{\alpha\beta} = -a_{\alpha\beta}^\perp.
\end{aligned}$$

(b) Since the map $\widehat{\Pi}|_{\widehat{Y}}: \widehat{Y} \rightarrow X^\perp$ is an isomorphism, $\widehat{\Pi}|_{\widehat{\Omega}}: \widehat{\Omega} \rightarrow \Omega^\perp$ and $\widehat{\Pi}|_{\widehat{Y}^0}: \widehat{Y}^0 \rightarrow X_0^\perp$ are isomorphisms.

(c) By Proposition 6.3, for any $\mathbf{v} \in X^\perp$

$$h^\perp(\mathbf{v}, \mathbf{v}) = h|_Y \circ \Pi^{-1}(\mathbf{v}, \mathbf{v}) + \bar{g}_{\alpha\beta} c^\alpha c^\beta.$$

If $\mathbf{v} \in X_0^\perp$, we have $\mathbf{v} = \Pi \mathbf{u} = (u_i - \lambda_{\alpha i} w_i^\alpha)$, $\mathbf{u} \in Y^0$. Then by Lemma 6.2, $c^\alpha = I_i^\alpha(\mathbf{u}_i, \mathbf{u}_{i+1}) = 0$. This implies the first equation in (c). To prove the second it is sufficient to note that $h = \hat{h} \circ \pi_\Gamma$ and $\Pi = \hat{\pi} \circ \Pi_\Gamma$.

(d) By (2.10),

$$\begin{aligned}
h(\widehat{\mathbf{q}}_\alpha, \mathbf{u}) &= \sum_i \langle A_i \hat{q}_{\alpha i} - B_i^* \hat{q}_{\alpha i+1} - B_{i-1} \hat{q}_{\alpha i-1}, u_i \rangle \\
&= \sum_i \langle -A_i \nu_{\alpha\beta i} w_i^\beta + B_i^* \nu_{\alpha\beta i+1} w_{i+1}^\beta + B_{i-1} \nu_{\alpha\beta i-1} w_{i-1}^\beta, u_i \rangle \\
&= \sum_i \langle B_i^* \Delta \nu_{\alpha\beta i} w_{i+1}^\beta - B_{i-1} \Delta \nu_{\alpha\beta i-1} w_{i-1}^\beta, u_i \rangle \\
&= s_{\alpha\gamma} \kappa^{\gamma\delta} \sum_i \langle B_i^* g_{\delta\beta i} w_{i+1}^\beta - B_{i-1} g_{\delta\beta i-1} w_{i-1}^\beta, u_i \rangle = -s_{\alpha\gamma} \kappa^{\gamma\delta} \sum_i \langle d_{\delta i}, u_i \rangle,
\end{aligned}$$

where the $d_{\delta i}$ are defined by (3.25). Proposition 3.9 implies the first assertion in (d). Analogously,

$$\begin{aligned}
h(\widehat{\mathbf{q}}_\alpha, \widehat{\mathbf{q}}_\beta) &= s_{\alpha\gamma} \kappa^{\gamma\delta} \sum_i \langle B_i^* g_{\delta\epsilon i} w_{i+1}^\epsilon - B_{i-1} g_{\delta\epsilon i-1} w_{i-1}^\epsilon, \widehat{q}_{\beta i} \rangle \\
&= s_{\alpha\gamma} \kappa^{\gamma\delta} \sum_i g_{\delta\epsilon i} (\langle B_i \widehat{q}_{\beta i}, w_{i+1}^\epsilon \rangle - \langle B_i w_i^\epsilon, \widehat{q}_{\beta i+1} \rangle) \\
&= -s_{\alpha\gamma} \kappa^{\gamma\delta} \bar{g}_{\delta\epsilon} c_\beta^\epsilon = -s_{\alpha\gamma} c_\beta^\gamma = a_{\alpha\beta}.
\end{aligned}$$

This implies the second assertion in (d).

(e) We get

$$\begin{aligned}
h^\perp(\mathbf{q}_\alpha^\perp, \mathbf{u}^\perp) &= \sum_i \langle (A_i - C_i)q_{\alpha i}^\perp - B_i^* q_{\alpha i+1}^\perp - B_{i-1} q_{\alpha i-1}^\perp, u_i^\perp \rangle \\
&= - \sum_i \langle C_i q_{\alpha i}^\perp + \lambda_{\alpha\beta i}^\perp A_i w_i^\beta - \lambda_{\alpha\beta i+1}^\perp B_i^* w_{i+1}^\beta - \lambda_{\alpha\beta i-1}^\perp B_{i-1} w_{i-1}^\beta, u_i^\perp \rangle \\
&= \sum_i g_{\gamma\delta i} (\langle B_i w_i^\gamma, q_{\alpha i+1} \rangle - \langle B_i q_{\alpha i}, w_{i+1}^\gamma \rangle) \langle B_i u_i^\perp, w_{i+1}^\delta \rangle \\
&= \sum_i g_{\alpha\delta i} \langle B_i u_i^\perp, w_{i+1}^\delta \rangle.
\end{aligned}$$

Since $u_i^\perp = u_i - g_{\varepsilon\beta i-1} \langle B_{i-1} w_{i-1}^\varepsilon, u_i \rangle v_i^\beta$, we obtain

$$g_{\gamma\delta i} \langle B_i u_i^\perp, w_{i+1}^\delta \rangle = - \langle d_{\gamma i}, u_i \rangle = 0.$$

This implies the first assertion in (e). Analogously,

$$\begin{aligned}
h^\perp(\mathbf{q}_\alpha^\perp, \mathbf{q}_\beta^\perp) &= \sum_i (g_{\alpha\delta i} \langle B_i q_{\beta i}, w_{i+1}^\delta \rangle - g_{\alpha\delta i-1} \langle B_{i-1} w_{i-1}^\delta, q_{\beta i} \rangle) \\
&= \sum_i (-g_{\alpha\beta i} + g_{\alpha\delta i} \langle B_i w_i^\delta, q_{\beta i+1} \rangle - g_{\alpha\delta i-1} \langle B_{i-1} w_{i-1}^\delta, q_{\beta i} \rangle) \\
&= -\bar{g}_{\alpha\beta} + s_{\alpha\beta}.
\end{aligned}$$

This implies the second assertion in (e).

A.2 Proof of Theorem 6.4

(a) For any constant vector λ^β we have $I^\alpha(\lambda^\beta \hat{\eta}_\beta, D\lambda^\beta \hat{\eta}_\beta) = \hat{c}_\beta^\alpha \lambda^\beta$. therefore for any constant vector r^α the coefficients λ^β can be chosen so that $I^\alpha(\lambda^\beta \hat{\eta}_\beta, D\lambda^\beta \hat{\eta}_\beta) = r^\alpha$. This implies the first equation in (a).

To prove the second equation in (a), we show that $\dim \Omega^\perp = k$ and $\Omega^\perp \cap X_0^\perp = 0$. In view of equation (6.24), it is sufficient to check that the matrix

$$e_{\alpha\beta} = \int_0^\tau g_{\alpha\delta} (D^\perp \zeta^\delta, \eta_\beta^\perp) dt$$

is nondegenerate. We have:

$$\begin{aligned}
e_{\alpha\beta} &= \int_0^\tau g_{\alpha\delta} (D\zeta^\delta, \eta_\beta^\perp) dt = \int_0^\tau g_{\alpha\delta} (D\zeta^\delta, \eta_\beta - \lambda_{\beta\varepsilon}^\perp \zeta^\varepsilon) dt \\
&= \int_0^\tau (g_{\alpha\delta} (D\zeta^\delta, \eta_\beta) - g_{\alpha\delta} (\eta_\beta, \zeta^\nu) g_{\nu\varepsilon} (D\zeta^\delta, \zeta^\varepsilon)) dt.
\end{aligned}$$

Using the equation

$$0 = \frac{d}{dt}(g_{\varepsilon\nu}g^{\nu\delta}) = (\dot{g}_{\varepsilon\nu}\zeta^\nu + 2g_{\varepsilon\nu}D\zeta^\nu, \zeta^\delta), \quad (\text{A.1})$$

we continue:

$$e_{\alpha\beta} = \frac{1}{2} \int_0^\tau \left(\frac{d}{dt}(g_{\alpha\delta}(\zeta^\delta, \eta_\beta)) - g_{\alpha\beta} \right) dt = \frac{1}{2} (s_{\alpha\beta} - \bar{g}_{\alpha\beta}) = -\frac{1}{2} a_{\alpha\beta}^\perp.$$

(b) The maps $\widehat{\Pi}|_{\widehat{\Omega}}: \widehat{\Omega} \rightarrow \Omega^\perp$ and $\widehat{\Pi}|_{\widehat{Y}^0}: \widehat{Y}^0 \rightarrow X_0^\perp$ are isomorphisms because $\widehat{\Pi}: \widehat{Y} \rightarrow X^\perp$ is an isomorphism.

(c) By Proposition 6.3, for any $\eta \in X^\perp$

$$h^\perp(\eta, \eta) = h|_Y \circ \Pi^{-1}(\eta, \eta) + \bar{g}_{\alpha\beta} c^\alpha c^\beta.$$

If $\eta \in X_0^\perp$, we have $\eta = \Pi\xi = \xi - \lambda_\alpha \zeta^\alpha$, $\xi \in Y^0$. Then by Lemma 6.1, $c^\alpha = I^\alpha(\xi, D\xi) = 0$. This implies the first equation in (c). To prove the second it is sufficient to note that $h = \hat{h} \circ \pi_\Gamma$ and $\Pi = \widehat{\Pi} \circ \pi_\Gamma$.

(d) Integrating by parts we get

$$\begin{aligned} h(\hat{\eta}_\alpha, \xi) &= \int_0^\tau (-D^2\hat{\eta}_\alpha + U\hat{\eta}_\alpha, \xi) dt = s_{\alpha\delta} \kappa^{\delta\varepsilon} \int_0^\tau (\dot{g}_{\varepsilon\nu}\zeta^\nu + 2g_{\varepsilon\nu}D\zeta^\nu, \xi) dt \\ &= s_{\alpha\delta} \kappa^{\delta\varepsilon} \int_0^\tau \frac{d}{dt}(g_{\varepsilon\nu}(\zeta^\nu, \xi)) dt = 0. \end{aligned}$$

Analogously,

$$h(\hat{\eta}_\alpha, \hat{\eta}_\beta) = s_{\alpha\delta} \kappa^{\delta\varepsilon} \int_0^\tau (\dot{g}_{\varepsilon\nu}\zeta^\nu + 2g_{\varepsilon\nu}D\zeta^\nu, \hat{\eta}_\beta) dt.$$

We define $\mu_{\alpha\beta}$ so that $\dot{\mu}_{\alpha\beta} = c_\alpha^\delta g_{\delta\beta}$. Then

$$(\zeta^\alpha, D(\hat{\eta}_\beta - \mu_{\beta\delta}\zeta^\delta)) - (\hat{\eta}_\beta - \mu_{\beta\delta}\zeta^\delta, D\zeta^\alpha) = c_\beta^\alpha - \dot{\mu}_{\beta\delta} g^{\delta\alpha} = 0.$$

Therefore,

$$\begin{aligned} \int_0^\tau (\dot{g}_{\varepsilon\nu}\zeta^\nu + 2g_{\varepsilon\nu}D\zeta^\nu, \hat{\eta}_\beta) dt &= \int_0^\tau \frac{d}{dt}(g_{\varepsilon\nu}(\zeta^\nu, \hat{\eta}_\beta - \mu_{\beta\delta}\zeta^\delta)) dt \\ &= - \int_0^\tau \mu_{\varepsilon\beta} dt = -\bar{g}_{\varepsilon\delta} c_\beta^\delta, \end{aligned}$$

where we have used (A.1). Finally, $\hat{h}(\hat{\eta}_\alpha, \hat{\eta}_\beta) = -s_{\alpha\delta} \kappa^{\delta\varepsilon} \bar{g}_{\varepsilon\nu} c_\beta^\nu = -s_{\alpha\delta} c_\beta^\delta = a_{\alpha\beta}$.

(e) Integrating by parts we get

$$h^\perp(\eta_\alpha^\perp, \xi^\perp) = \int_0^\tau \left(((-D\Pi D + U)\eta_\alpha^\perp, \xi^\perp) - 3g_{\delta\varepsilon}(\eta_\alpha^\perp, D\zeta_\delta)(\xi^\perp, D\zeta_\varepsilon) \right) dt.$$

Direct, but lengthy computation gives

$$h^\perp(\eta_\alpha^\perp, \xi^\perp) = \int_0^\tau 2g_{\alpha\varepsilon}(\xi^\perp, D\zeta^\varepsilon) dt = \int_0^\tau \frac{d}{dt}(g_{\alpha\varepsilon}(\xi, \zeta^\varepsilon)) dt = 0.$$

Here we have used (A.1). Analogously,

$$\begin{aligned} h^\perp(\eta_\alpha^\perp, \eta_\beta^\perp) &= \int_0^\tau 2g_{\alpha\varepsilon}(D\zeta^\varepsilon, \eta_\beta^\perp) dt = \int_0^\tau 2g_{\alpha\varepsilon}((D\zeta^\varepsilon, \eta_\beta) - (D\zeta^\varepsilon, \zeta^\nu)g_{\nu\vartheta}(\zeta^\vartheta, \eta_\beta)) dt \\ &= \int_0^\tau \left(-g_{\alpha\beta} + \frac{d}{dt}(g_{\alpha\vartheta}(\zeta^\vartheta, \eta_\beta)) \right) dt = -\bar{g}_{\alpha\beta} + s_{\alpha\beta}. \end{aligned}$$

A.3 Degenerate case

In this subsection we consider the case when the nondegeneracy assumption A' on p. 56 fails, that is, $\text{rank}(\zeta^1(t), \dots, \zeta^k(t))$ drops on $\Sigma \subset \mathbb{R}/\tau\mathbb{Z}$. We will see that the Routh reduction of the system (E, Λ) to (E^\perp, Λ^\perp) and other results on elimination of degeneracy hold with minor modifications of the proofs. Note that for DLS condition A, which is similar to A', is probably necessary.

Lemma A.1. *The family $(F_t)_{t \notin \Sigma}$ can be extended to a smooth vector bundle $F = (F_t)_{t \in \mathbb{R}/\tau\mathbb{Z}}$. Thus the orthogonal complement $E^\perp = (E_t^\perp)$ is a smooth vector bundle. The operator D^\perp on E^\perp and the reduced Lagrangian Λ^\perp defined for $t \notin \Sigma$ can be smoothly extended to $t \in \Sigma$.*

Proof. Suppose that $0 \in \Sigma$, and let

$$\text{rank}(\zeta^1(0), \dots, \zeta^k(0)) = k - l.$$

Without loss of generality we may assume that

$$\zeta^1(0) = \dots = \zeta^l(0) = 0, \quad \text{rank}(\zeta^{l+1}(0), \dots, \zeta^k(0)) = k - l.$$

Then $\text{rank}(D\zeta^1(0), \dots, D\zeta^k(0)) = l$, or else the solutions ζ^1, \dots, ζ^l of the variational system are dependent. Since

$$I^\alpha(\zeta^\beta, D\zeta^\beta) = (\zeta^\alpha, D\zeta^\beta) - (\zeta^\beta, D\zeta^\alpha) = 0,$$

we have

$$(D\zeta^\alpha(0), \zeta^\beta(0)) = 0, \quad \alpha = 1, \dots, l, \quad \beta = l+1, \dots, k.$$

Thus

$$\text{rank}(D\zeta^1(0), \dots, D\zeta^l(0), \zeta^{l+1}(0), \dots, \zeta^k(0)) = k.$$

Since

$$D(\zeta^\alpha - tD\zeta^\alpha) = -tD^2\zeta^\alpha = -tU\zeta^\alpha = O(t^2),$$

we have

$$\zeta^\alpha(t) = tD\zeta^\alpha(t) + O(t^3), \quad \alpha = 1, \dots, l.$$

Thus, the space

$$F_t = \text{span}\left(\frac{1}{t}D\zeta^1(t), \dots, \frac{1}{t}D\zeta^l(t), \zeta^{l+1}(t), \dots, \zeta^k(t)\right), \quad t \neq 0,$$

has a limit

$$F_0 = \text{span}(D\zeta^1(0), \dots, D\zeta^l(0), \zeta^{l+1}(0), \dots, \zeta^k(0))$$

as $t \rightarrow 0$, and F_t is smooth at $t = 0$. The first statement is proved.

Since E_t^\perp is smooth at $t = 0$, also $\Pi_t: E_t \rightarrow E_t^\perp$ is smooth, and hence the operator D^\perp is smooth. Finally we need to check that the term

$$(C\eta, \eta) = g_{\alpha\beta}(u, D^\perp\zeta^\alpha)(u, D^\perp\zeta^\beta)$$

in Λ^\perp is smooth at $t = 0$.

Denote $j^{\alpha\beta} = (D\zeta^\alpha(0), D\zeta^\beta(0))$, $\alpha, \beta = 1, \dots, l$. Then $(j^{\alpha\beta})$ is a nondegenerate matrix and

$$(D\zeta^\alpha, D\zeta^\beta) = j^{\alpha\beta} + O(t^2).$$

Thus

$$g^{\alpha\beta} = (\zeta^\alpha, \zeta^\beta) = t^2(D\zeta^\alpha, D\zeta^\beta) + O(t^4) = t^2j^{\alpha\beta} + O(t^4), \quad \alpha, \beta = 1, \dots, l.$$

The matrix $(g^{\alpha\beta}(0))$, $\alpha, \beta = l+1, \dots, k$, is nondegenerate, while one can show that

$$g^{\alpha\beta} = O(t^2), \quad \alpha = 1, \dots, l, \quad \beta = l+1, \dots, k.$$

Thus for the inverse matrix $(g_{\alpha\beta})$ we obtain

$$g_{\alpha\beta} = t^{-2}(j_{\alpha\beta} + O(t^2)), \quad \alpha, \beta = 1, \dots, l.$$

The block $(g_{\alpha\beta})$, $\alpha, \beta = l+1, \dots, k$, is smooth and nondegenerate and the block $(g_{\alpha\beta})$, $\alpha = 1, \dots, l$, $\beta = l+1, \dots, k$, is smooth.

Since $D^\perp\zeta^\alpha(t) = O(t^2)$ for $\alpha = 1, \dots, l$, we obtain that C is smooth at $t = 0$. Thus the reduced Lagrangian Λ^\perp is smooth on E^\perp . \square

In fact, everything we have done in §6 holds in the singular case. For example, let us check that the projection $\Phi: X \rightarrow \widehat{Y}$ along Z is well defined and smooth. As in the nondegenerate case, we have

$$Z = \{\xi \in X : \xi(t) \in F_t\} = \left\{ \xi(t) = \lambda_\alpha(t)\zeta^\alpha(t) : \int_0^\tau g^{\alpha\beta}\dot{\lambda}_\alpha\dot{\lambda}_\beta dt < \infty \right\},$$

but now λ_α may be singular for $t \in \Sigma$.

Take $\eta \in X$ and look for $\lambda_\alpha(t)$ such that $\xi = \eta + \lambda_\alpha \zeta^\alpha \in Y$. Then by (6.8) and (6.23),

$$\dot{\lambda}_\alpha = g_{\alpha\beta} c^\beta - g_{\alpha\beta} I^\beta(\eta, D\eta) = g_{\alpha\beta} c^\beta + \frac{d}{dt}(g_{\alpha\beta}(\eta, \zeta^\beta)) - 2g_{\alpha\beta}(\eta, D^\perp \zeta^\beta).$$

The last term is smooth for $t \in \Sigma$. Suppose again that $0 \in \Sigma$. Then we obtain

$$\lambda_\alpha(t) = -\frac{1}{t} \sum_{\beta=1}^l j_{\alpha\beta} c^\beta + \frac{1}{t} \sum_{\beta=1}^l j_{\alpha\beta}(\eta, \zeta^\beta) + \text{smooth terms}, \quad \alpha = 1, \dots, l.$$

It follows that $\lambda_\alpha(t)\zeta^\alpha(t)$ is smooth, so $\xi(t)$ is smooth.

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